Reverse Mathematics and Arithmetic Transfinite Recursion.

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What axioms are necessary to do mathematics?

- Is the fifth postulate necessary for Euclidean geometry?
- Is Peano Arithmetic enough to prove all the true statements about the natural numbers?
- Which large cardinals can be proved to exist in ZFC?
Simpson and Friedman’s program of Reverse Mathematic deals with Second Order Arithmetic ($\mathcal{Z}_2$).

- In $\mathcal{Z}_2$ we can talk about finite and countable objects.
- $\mathcal{Z}_2$ is much weaker than ZFC,
- and its much stronger than PA.
- In $\mathcal{Z}_2$ one can talk about
  - Countable algebra,
  - (non-set theoretic) combinatorics,
  - Real numbers,
  - Manifolds, continuous functions, differential equations...
  - Complete separable metric spaces.
  - Logic, computability theory,...
  - ....
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Fix a base theory.  
(We use RCA\(_0\) that essentially says that the computable sets exists)

Pick a theorem \(T\).

What axioms do we need to add to RCA\(_0\) to prove \(T\).

Suppose we found axioms \(A_0, ..., A_k\) of \(\mathcal{Z}_2\) such that

\[
\text{RCA}_0 \text{ proves } A_0 \land ... \land A_k \Rightarrow T.
\]

How do we know these are necessary?

It’s often the case that RCA\(_0\) also proves \(T \Rightarrow A_0 \land ... \land A_k\)

Then, we know that RCA\(_0 + A_0, ..., A_k\) is the least system (extending RCA\(_0\)) where \(T\) can be proved.
Main question revisited

1. Fix a base theory.  
   (We use RCA$_0$ that essentially says that the computable sets exists)

2. Pick a theorem $T$.

3. What axioms do we need to add to RCA$_0$ to prove $T$.

4. Suppose we found axioms $A_0, \ldots, A_k$ of $\mathbb{Z}_2$ such that
   
   $\text{RCA}_0$ proves $A_0 \land \ldots \land A_k \Rightarrow T$.

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The following are equivalent over RCA$_0$:

- (Weak König’s lemma) Every infinite binary tree has an infinite path.
- Every open covering of $[0, 1] \subset \mathbb{R}$ has a finite subcovering.
- Every countable commutative ring has a prime ideal.

TFAE over RCA$_0$, and stronger than the ones above:

- (Arithmetic Comprehension)
  Every formula without set quantifiers defines a set.
- Every bounded sequence of real number has a convergent subsequence.
- Every countable commutative ring has a maximal ideal.
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The language for $\mathcal{Z}_2$ has:

- $0, 1, +, \times, \leq, \in$.
- Logical symbols $\&$, $\lor$, $\neg$.
- Variables $x, y, z$... used for natural numbers, and quantification over these $\forall x$... and $\exists x$...
- Variables $X, Y, Z$... used for sets of natural numbers, and quantification over these $\forall X$... and $\exists X$...

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Every countable object can be encoded as a subset of \( \mathbb{N} \).
Example of coding

Finite tuples of natural numbers can be encoded by one natural number. (Example, encode \( \langle x, y \rangle \) by \((x + y)^2 + x\).) So, we can deal with \( \mathbb{Z} \) and \( \mathbb{Q} \).

Every finite object can be encoded by a sequence of 0s and 1s, and hence by a single natural number.

So, a subset of \( \mathbb{N}^3 \) can be encoded as a subset of \( \mathbb{N} \).

**Example:** A countable ring \( A = (A, 0, 1, +_A, \times_A) \) can be encoded by three sets \( A \subseteq \mathbb{N} \), \( +_A \subseteq \mathbb{N}^3 \) and \( \times_A \subseteq \mathbb{N}^3 \).
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Axioms of $\mathcal{Z}_2$

**Semi-ring Axioms:** $\mathbb{N}$ is an ordered semi-ring.

**Induction Axioms:** For every formula $\varphi(n)$, $IND(\varphi)$

$$(\varphi(0) \& \forall n(\varphi(n) \Rightarrow \varphi(n+1))) \Rightarrow \forall n \varphi(n)$$

**Comprehension Axioms:** For every formula $\varphi(n)$

$CA(\varphi)$

$$\exists X \forall n \ (n \in X \Leftrightarrow \varphi(n))$$

**$RCA_0$** consists of: Semi-ring Axioms + $\Sigma^0_1$-IND + $\Delta^0_1$-CA.

$\Delta^0_1$-CA says that for every computer program $p$ there is a set $X$ such that

$$n \in X \Leftrightarrow p(n) = yes$$

(where $p$ can use information from sets that we know exist)
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The big five

\[ \text{\( \Pi^1_1\)-CA}_0 \]
\[ \text{ATR}_0 \]
\[ \text{ACA}_0 \]
\[ \text{WKL}_0 \]
\[ \text{RCA}_0 \]
ACA₀

**Def:**
A formula is *arithmetic* if it has no quantifiers over sets.

ACA₀ is RCA₀ + Arithmetic comprehension.

where *Arithmetic comprehension* is the scheme of axioms

\[ \exists X \forall n \ (n \in X \iff \varphi(n)) \]

where \( \varphi \) is any arithmetic formula.

**Theorem**
The following are equivalent over RCA₀:

- ACA₀
- Every countable vector space has a basis.
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**WKL\textsubscript{0}**

\textit{WKL}\textsubscript{0} is RCA\textsubscript{0} + Weak K"{o}nig’s lemma.

\textit{Weak K"{o}nig’s lemma} says: Every infinite subtree of the full binary tree has an infinite path.

**Theorem**

The following are equivalent over RCA\textsubscript{0}:

- WKL\textsubscript{0}
- Every continuous function on [0, 1] is uniformly continuous.
The systems in higher end

$WKL_0$ is $RCA_0 +$ Weak König’s lemma.

*Weak König’s lemma* says: Every infinite subtree of the full binary tree has an infinite path.

**Theorem**

The following are equivalent over $RCA_0$:

- $WKL_0$
- Every continuous function on $[0, 1]$ is uniformly continuous.
**Π\(^1_1\)-separation**

**Def:** A formula is Π\(^1_1\) if it has the form \(\forall X \psi\), where \(\psi\) is an arithmetic formula.

Π\(^1_1\)-CA\(_0\) is RCA+ Π\(^1_1\) comprehension.

where Π\(^1_1\) **comprehension** is the scheme of axioms

\[\text{CA}(\varphi) \equiv \exists X \forall n \ (n \in X \iff \varphi(n))\]

where \(\varphi\) is any Π\(^1_1\) formula.

**Theorem**

The following are equivalent over RCA\(_0\):

- Π\(^1_1\)-CA\(_0\)
- Every countable Abelian group is a direct sum of a divisible group and a reduced group.

**Def:** A group \(G\) is divisible if \(\forall a \in G \forall n \in \mathbb{N} \exists b (nb = a)\).

**Def:** A group \(G\) is reduced if it has no divisible subgroup.
**Π₁⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻⁻ indeb
**ATR**

**ATR** \(_0\) is RCA\(_0\) + Arithmetic Transfinite Recursion.

[Simpson] **ATR** \(_0\) is the least system where one can develop a reasonable theory of ordinals.

**Theorem**

The following are equivalent over RCA\(_0\):

- ATR\(_0\)
- Given two ordinals, one is an initial segment of the other one.

**Def:** An *ordinal* is a linear ordering without descending sequences. Examples: 1, 2, 3, ..., \(\omega\), \(\omega + 1\), ..., \(\omega + \omega\), ..., \(\omega \times \omega\), ...
Reduced $p$-groups

**Def:** A group $G$ is *divisible* $\forall a \in G \ \forall n \in \mathbb{N} \exists b \ (nb = a)$.

**Def:** A group $G$ is *reduced* if it has no divisible subgroup.

**Theorem**

*TFAE over RCA$_0$*

- ATR$_0$
- \[Friedman, Simpson, Smith '77\] Every reduced $p$-group has an Ulm decomposition.
- \[Friedman '01; M, Grenberg '05\] The reduced $p$-groups are well-quasi-ordered by embeddability. *That is: there is no infinite descending sequence and no infinite antichain.*

We want to claim:

ATR$_0$ is the least system where one can develop a reasonable theory of reduced-$p$-groups.
Reduced $p$-groups

We pick two very basic statements about reduced-$p$-groups.

$\exists$-$ISO(\mathcal{R}-p\mathcal{G})$
If $\langle G_n \rangle_{n \in \mathbb{N}}$ is a sequence of reduced-$p$-groups, there exists a set $X \subseteq \mathbb{N}^2$ such that $\langle n, m \rangle \in X \iff G_n \cong G_m$

$\exists$-$EMB(\mathcal{R}-p\mathcal{G})$
If $\langle G_n \rangle_{n \in \mathbb{N}}$ is a sequence of reduced-$p$-groups, there exists a set $X \subseteq \mathbb{N}^2$ such that $\langle n, m \rangle \in X \iff G_n$ embeds in $G_m$

Theorem

TFAE over RCA$_0$

- ATR$_0$
- $\exists$-$ISO(\mathcal{R}-p\mathcal{G})$ [Shore, Solomon; M, Grenberg '05]
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**Theorem**

\text{TFAE over } RCA_0

- $ATR_0$
- $\exists \text{-ISO}(\mathcal{R-p-G})$ [Shore, Solomon; M, Grenberg '05]
- $\exists \text{-EMB}(\mathcal{R-p-G})$ [M, Grenberg '05]
Super atomic Boolean algebras

**Def:** A Boolean alg. $B$ is *atomless* if $\forall x \in B \setminus \{0\} \exists y (0 < y < x)$

**Def:** $B$ is *superatomic* if it has no atomless subalgebra.

**Theorem**

[M, Grenberg ’05] TFAE over RCA$_0$

- $\text{ATR}_0$;
- $\exists\text{-ISO}(\text{SABA}); \forall \langle B_n \rangle_{n \in \mathbb{N}} \exists X \subseteq \mathbb{N}^2 \langle n, m \rangle \in X \Leftrightarrow B_n \cong B_m$
- $\exists\text{-EMB}(\text{SABA})$;
- For any two superatomic BAs, one embeds into the other one;
- If two superatomic BAs can be embedded into each other, then they are isomorphic;
- The superatomic BAs are well-quasi-ordered by embeddability.

$\text{ATR}_0$ is the least system where one can develop a reasonable theory of superatomic Boolean algebras.
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[Ranjan, Grenberg '05] TFAE over RCA

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**Def:** A Boolean alg. $B$ is *atomless* if $\forall x \in B \setminus \{0\} \exists y (0 < y < x)$

**Def:** $B$ is *superatomic* if it has no atomless subalgebra.

**Theorem**

[M, Grenberg '05] **TFAE over RCA$_0$**

- $\text{ATR}_0$;
- $\exists$-$\text{ISO}(\text{SABA})$; $\forall\langle B_n \rangle_{n \in \mathbb{N}} \exists X \subseteq \mathbb{N}^2 \langle n, m \rangle \in X \iff B_n \cong B_m$
- $\exists$-$\text{EMB}(\text{SABA})$;
- For any two superatomic BAs, one embeds into the other one;
- If two superatomic BAs can be embedded into each other, then they are isomorphic;
- The superatomic BAs are well-quasi-ordered by embeddability.

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Reverse Mathematics and Arithmetic Transfinite Recursion.
Countable Compact Metric spaces

Let \( CCMS \) be the class of Countable Compact Metric Spaces.

**Theorem**

[\( M, \; \text{Grenberg '05} \)] TFAE over \( RCA_0 \)

- \( ATR_0 \);
- \( \exists \text{-ISO}(CCMS) \);
- \( \exists \text{-EMB}(CCMS) \);
- For any two \( CCMS \), one embeds into the other one;
  [\( \text{Friedman, Hirst '91} \)]
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Well founded Trees

Let $\mathcal{WFT}$ be the class of Well-Founded trees, that is, trees with no infinite paths.

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Theorem [Fraïssé’s Conjecture ’48; Laver ’71]

FRA: The countable linear orderings form a WQO with respect to embeddability. (i.e., there are no infinite descending sequences and no infinite antichains.)

Theorem [Shore ’93]

FRA implies ATR\(_0\) over RCA\(_0\).

Conjecture: [Clote ’90][Simpson ’99][Marcone]

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Fraïssé’s Conjecture

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Fraïssé’s conjecture.

\( \text{RCA}_0 + \) Fraïssé’s conjecture is a \textit{Robust} system.

**Theorem**

The following are equivalent over \( \text{RCA}_0 \)

- \( \text{FRA} \);
- Every scattered linear ordering can be written as a finite sum of indecomposable ones;
- Every indecomposable linear ordering can be written either as an \( \omega \)-sum or as an \( \omega^* \) sum of indecomposable l.o. of smaller rank.
- Well founded trees, labeled with \{+,-\} are well-quasi-ordered.
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RCA$_0$ + Fraïssé’s conjecture is a *Robust* system.

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A Partition theorem

**Theorem:** [Folklore] If we color \( \mathbb{Q} \) with finitely many colors, there exists an embedding \( \mathbb{Q} \rightarrow \mathbb{Q} \) whose image has only one color.

**Theorem:** [Laver ’72]
For every ctable \( \mathcal{L} \), there exists \( n \in \mathbb{N} \), such that:
if \( \mathcal{L} \) is colored with finitely many colors, there is an embedding \( \mathcal{L} \rightarrow \mathcal{L} \) whose image has at most \( n \) many colors.

**Theorem**
\( \text{FRA is implied by Laver’s Theorem above over } \text{RCA}_0. \)

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A Partition theorem

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**Theorem**

*FRA is implied by Laver’s Theorem above over RCA$_0$.*

**Conjecture**

*FRA is equivalent to Laver’s Theorem above over RCA$_0$.***
The indecomposability statement

- $\mathcal{L}$ is **scattered** if $\mathbb{Q} \not\preceq \mathcal{L}$.
- $\mathcal{L}$ is **indecomposable** if whenever $\mathcal{L} = \mathcal{A} + \mathcal{B}$, either $\mathcal{L} \preceq \mathcal{A}$ or $\mathcal{L} \preceq \mathcal{B}$.
- $\mathcal{L}$ is **indecomposable to the right** if for every non-trivial cut $\mathcal{L} = \mathcal{A} + \mathcal{B}$, we have $\mathcal{L} \preceq \mathcal{B}$.
- $\mathcal{L}$ is **indecomposable to the left** if for every non-trivial cut $\mathcal{L} = \mathcal{A} + \mathcal{B}$, we have $\mathcal{L} \preceq \mathcal{A}$.

**Theorem [Jullien ‘69]** INDEC: Every scattered indecomposable linear ordering is indecomposable either to the right or to the left.

**Theorem ([M 05])**

*INDEC is strictly in between ACA$_0$ and ATR$_0$.***
The indecomposability statement

- \( \mathcal{L} \) is scattered if \( \mathbb{Q} \not\preceq \mathcal{L} \).
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- \( \mathcal{L} \) is indecomposable to the right if for every non-trivial cut \( \mathcal{L} = A + B \), we have \( \mathcal{L} \preceq B \).
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Consider \( HYP = \{ \) hyperarithmetic sets\} \\
= \{ \) sets definable with a 1st order computable infinitary formula \} \\
Think of it as an \( \omega \)-model of second order arithmetic.

Theories that have \( HYP \) as their least \( \omega \)-models have been studied since the seventies. 

**Examples:** \( \Delta^1_1\)-CA\(_0\), \( \Sigma^1_1\)-AC\(_0\), \( \Sigma^1_1\)-DC\(_0\) and weak-\( \Sigma^1_1\)-AC\(_0\).

**Definition:** We say that a sentence \( S \) is a sentence of hyperarithmetic analysis if the least \( \omega \)-model of \( RCA_0 + S \) is \( HYP \).

**Theorem**

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\( \text{INDEC} \) is a sentence of hyperarithmetic analysis.
Finitely terminating games

To each well founded tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$, we associate a game $G(T)$ which is played as follows. Player I starts by playing a number $a_0 \in \mathbb{N}$ such that $\langle a_0 \rangle \in T$. Then player II plays $a_1 \in \mathbb{N}$ such that $\langle a_0, a_1 \rangle \in T$, and then player I plays $a_2 \in \mathbb{N}$ such that $\langle a_0, a_1, a_2 \rangle \in T$. They continue like this until they get stuck. The first one who cannot play loses.

We will refer to games of the form $G(T)$, for $T$ well-founded, as finitely terminating games.

Observation Finitely terminating games are in 1-1 correspondence with clopen games.
Finitely terminating games

Let $T_I = \{ \sigma \in T : |\sigma| \text{ is even} \}$, $T_{II} = \{ \sigma \in T : |\sigma| \text{ is odd} \}$. A strategy for $I$ in $G(T)$ is a function $s : T_I \to \mathbb{N}$. A strategy $s$ for $I$ is a winning strategy if whenever $I$ plays following the $s$, he wins. A game $G(T)$ is determined if there is a winning strategy for one of the two players.

We say that a game is completely determined if there is a map $d : T \to \{W, L\}$ such that for every $\sigma \in T$,

- $d(s) = W \iff I$ has a winning strategy in $G(T_\sigma)$, and
- $d(s) = L \iff II$ has a winning strategy in $G(T_\sigma)$.

Note that completely determined games are determined.
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Note that completely determined games are determined.
Known results

**Theorem** [Steel 1976] The following are equivalent over $\text{RCA}_0$.

- $\text{ATR}_0$;
- Every finitely terminating game is determined;
- Every finitely terminating game is completely determined.
New statements

- **CDG-CA**: Given a sequence \( \{ T_n : n \in \mathbb{N} \} \) of completely determined trees, there exists a set \( X \) such that
  \[ \forall n \ (n \in X \iff I \text{ has a winning strategy in } G(T_n)). \]

- **CDG-AC**: Given a sequence \( \{ T_n : n \in \mathbb{N} \} \) of completely determined trees, there exists a sequence \( \{ d_n : n \in \mathbb{N} \} \) where for each \( n \), \( d_n : T \to \{ W, L \} \) is a winning function for \( G(T_n) \).

- **DG-CA**: Given a sequence \( \{ T_n : n \in \mathbb{N} \} \) of determined trees, there exists a set \( X \) such that
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Introduction
The Main Five systems
The systems in higher end
Arithmetic Transfinite Recursion
Fraïssé’s Conjecture
Hyperarithmetic analysis

Implications between statements

Theorem

\[ \Sigma_1^1 - DC_0 \leadsto \Sigma_1^1 - AC_0 \]

\[ \Sigma_1^1 - AC_0 \downarrow \downarrow \ DG-AC \]

\[ DG-CA \Leftrightarrow \Delta_1^1 - CA_0 \]

weak \( \Sigma_1^1 - AC_0 \)

\[ CDG-AC \Leftrightarrow CDG-CA \]

\[ \downarrow \downarrow \ JI. \]

INDEC

over RCA_0.


Reverse Mathematics and Arithmetic Transfinite Recursion.