Lusternik-Schnirelmann category of Orbifolds

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Abstract
The idea is to generalize to the case of orbifolds the classic Lusternik-Schnirelmann theory. This paper defines a notion of LS-category for orbifolds. We show that some of the classical estimates for the regular category have their analogue in the case of orbifolds. We examine the topic in some detail using a mixture of approaches from equivariant theory and foliations.

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1 Orbifolds

The concept of orbifold was introduced by Satake [17] in 1956 under the name of V-manifold and developed later by Thurston [18] and Haefliger [10]. In this section we review some basic definitions concerning orbifolds.

Let $X$ be a Hausdorff space.

An orbifold chart for an open set $V \subset X$ is a triple $(\tilde{V}, G, \varphi)$ such that

1. $\tilde{V}$ is a connected open subset of $\mathbb{R}^n$,
2. $G$ is a finite subgroup of diffeomorphisms of $\tilde{V}$,
3. $\varphi: \tilde{V} \to V$ is a $G$-invariant map inducing a homeomorphism from $\tilde{V}/G$ onto $V$.

If $V_i \subset V_j$, an injection $(\lambda_{ij}, h_{ij}): (\tilde{V}_i, G_i, \varphi_i) \to (\tilde{V}_j, G_j, \varphi_j)$ between the orbifold charts is:

1. an embedding $\lambda_{ij}: \tilde{V}_i \to \tilde{V}_j$ such that $\varphi_j \circ \lambda_{ij} = \varphi_i$ and
2. an injective homomorphism $h_{ij}: G_i \to G_j$ with $\lambda_{ij}$ equivariant respect to $h_{ij}$, i.e. $\lambda_{ij}(gx) = h_{ij}(g)\lambda_{ij}(x)$ for all $g \in G_i$, $x \in \tilde{V}_i$.

An orbifold atlas on $X$ is a family $\{(\tilde{V}_i, G_i, \varphi_i)\}_{i \in I}$ of orbifold charts such that

1. $\mathcal{V} = \{V_i\}_{i \in I}$ is a covering of $X$
2. If $V_i \subset V_j$, then there exists an injection $(\lambda_{ij}, h_{ij}): (\tilde{V}_i, G_i, \varphi_i) \to (\tilde{V}_j, G_j, \varphi_j)$
3. For any point $x \in V_i \cap V_j$, with $V_i, V_j \in \mathcal{V}$, there exists $W \in \mathcal{V}$ such that $x \in W \subset V_i \cap V_j$. 

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The open sets $V_i \in \mathcal{V}$ are called \textit{basic} open sets.

Two atlases are \textit{equivalent} if they are contained in the same maximal atlas.

An orbifold $X$ of dimension $n$ is a Hausdorff space equipped with an equivalent class of orbifold atlases.

A manifold $M$ is a special case of orbifold where every group $G_i$ is the trivial group $G_i = \{1\}$ for all $i \in I$. This orbifold structure on $M$ will be called the \textit{trivial orbifold structure}.

A typical example of orbifold is provided by a properly discontinuous action of a group $G$ on a manifold $\tilde{\mathcal{M}}$. The quotient space $\tilde{\mathcal{M}}/G$ has an orbifold structure inherited from the manifold structure of $\tilde{\mathcal{M}}$. The orbifold charts are obtained by suitable restrictions of the quotient map $\tilde{\mathcal{M}} \to \tilde{\mathcal{M}}/G$. In this case, $\tilde{\mathcal{M}}/G$ is called a \textit{global} orbifold.

To each point $x$ in an orbifold $X$ we can associate a finite subgroup $G_x$ of $GL_n(\mathbb{R})$, well defined up to conjugacy. Let $V$ be an open basic neighborhood of $x$ and consider the corresponding chart $(\tilde{V}, G, \varphi)$.

We can identify $G$ and its image by $h$ since $h$ is a monomorphism, $h(G) = G$. If $g \in G_x \tilde{x}$ we will show that there is $\gamma \in H$ such that $\gamma h(g) \gamma^{-1} \in H_{\tilde{w}}$. Let $g \in G_x \tilde{x}$ and $\lambda(\tilde{x}) = \tilde{y}$. Then $g \tilde{x} = \tilde{x}$ and there exists $\gamma \in H$ such that $\tilde{w} = \gamma \tilde{y}$. We have:

$$h(g)\tilde{y} = h(g)\lambda(\tilde{x}) = \lambda(g \tilde{x}) = \lambda(\tilde{x}) = \tilde{y} \text{ and } \gamma h(g)\gamma^{-1} \tilde{w} = \gamma h(g)\gamma^{-1} \gamma \tilde{y} = \gamma h(g)\tilde{y} = \gamma \tilde{y} = \tilde{w}$$

Then the local group $G_x$ of $x$ is well defined up to conjugacy. We can think of $G_x$ as the conjugacy class $(G_x)_{\tilde{x}}$ of $G_x$ with $\varphi(\tilde{x}) = x$.

We will say that a point $x \in X$ is \textit{regular} if $G_x$ is trivial, and \textit{singular} otherwise. The set $\Sigma X = \{x \in X | G_x \neq \{1\}\}$ is the \textit{singular locus} of $X$. The singular locus is not an orbifold in general.

**PROPOSITION 1.1** [18] The singular locus $\Sigma X$ of an orbifold $X$ is a closed set with empty interior.

Observe that the set of regular points is a dense connected open subset of $X$.

**DEFINITION 1.2** We call a reduced orbifold if $G_x$ acts effectively for all $x \in X$.
Every reduced orbifold is a quotient of a smooth manifold by and effective almost free action of a Lie group $G$ [1]. If the group is discrete and the action is proper (for instance, if $G$ is finite) the orbifold is a global quotient.

1.1 Orbifold maps

There are many definitions of orbifold maps. For our purposes we choose to use essentially the original definition of Satake. For each point in the orbifold, this definition implies the existence of an homomorphism between the isotropy groups which will play a key role in our definition of category.

A map $f: X \to X'$ between orbifolds is a smooth orbifold map if for any point $x \in X$ there is a basic open set $V \subset X$ around $x$ and a basic open set $V' \subset X'$ around $f(x)$ with corresponding charts $(\tilde{V}, G, \varphi)$ and $(\tilde{V}', G', \varphi')$ verifying the following conditions:

1. there exists a smooth map $\tilde{f}: \tilde{V} \to \tilde{V}'$ and
2. an homomorphism $\bar{f}: G \to G'$

such that $\varphi' \tilde{f} = f \varphi$ and $\tilde{f}$ is equivariant with respect to $\bar{f}$:

$$\bar{f}(g x) = \bar{f}(g) \tilde{f}(x) \text{ for all } g \in G, x \in \tilde{V}.$$ 

An open set $U \subset X$ is an orbifold with the orbifold structure induced by the orbifold structure on $X$.

The inclusion map $i: U \to X$ is an orbifold map with $\tilde{f} = \text{id}: \tilde{V} \to \tilde{V}$ and $\tilde{f} = \text{id}: G \to G$. The constant map $c: X \to X'$ with $c(x) = x_0 \in X'$ $\forall x \in X$ is an orbifold map considering $\tilde{f}$ and $\bar{f}$ to be constant maps, that is $\tilde{f}: \tilde{V} \to \tilde{V}'$ such that $\tilde{f}(\tilde{x}) = \tilde{x}_0$ for all $\tilde{x} \in \tilde{X}$ where $\varphi(\tilde{x}) = x$ and $\varphi'(\tilde{x}_0) = x_0$ and $\tilde{f}: G \to G'$ such that $\bar{f}(g) = 1$ for all $g \in G$.

For any two orbifolds $X$ and $X'$, the cartesian product $X \times X'$ is an orbifold with the orbifold structure given by the product of the orbifolds structures of $X$ and $X'$. This structure on $X \times X'$ is called the product orbifold structure.

A homotopy $H: X \times I \to Y$ between orbifolds is an orbifold homotopy if $H$ is a smooth orbifold map considering $I$ with the trivial orbifold structure and $X \times I$ with the product orbifold structure.

2 Categorical sets for orbifolds

We describe an open subset $U \subset X$ as orbifold categorical if there is an orbifold homotopy $H: U \times I \to X$ such that $H_0: U \to X$ is inclusion and $H_1: U \to X$ is the constant map. Here $U$ is regarded as an orbifold with the orbifold structure induced by the one on $X$. In other words, the open subset $U$ of $X$ is orbifold categorical if the inclusion $U \hookrightarrow X$ factors through a point up to orbifold homotopy.
DEFINITION 2.1 The orbifold category \( \text{cat}_{\text{orb}} X \) of an orbifold \( X \) is the least number of orbifold categorical open sets required to cover \( X \). If no such covering exists we say that the orbifold category is infinite.

Note that in general, the orbifold category \( \text{cat}_{\text{orb}} X \) of an orbifold \( X \) does not coincide with the ordinary category \( \text{cat} X \) of the underlying topological space. In general, \( \text{cat} X \leq \text{cat}_{\text{orb}} X \).

The orbifold category is an invariant of orbifold homotopy type that coincides with the ordinary category when the orbifold structure is trivial.

If \( U \) is an orbifold categorical open set, let \( H: U \times I \to X \) be the orbifold homotopy. For any point \( (x, t) \in U \times I \), let \( y = H(x, t) \). There is a basic open set \( V_x \subset U \) around \( x \), an open set \( I_t \subset I \) around \( t \) and a basic open set \( V_y \subset X \) around \( y \) with \( G_x \) and \( G_y \) the corresponding isotropy groups. The homotopy \( H \) is locally lifted to a homotopy \( \tilde{H}_{(x, t)}: \tilde{V}_x \times I_t \to \tilde{V}_y \) which is equivariant with respect to the homomorphism \( \tilde{H}_{(x, t)}: G_x \to G_y \).

PROPOSITION 2.2 Let \( H: U \times I \to X \) be an orbifold homotopy of the inclusion. Then \( \tilde{H}_{(x, t)}: G_x \to G_y \) is injective for all \( x \in U \) and \( t \in I \) with \( y = H(x, t) \).

Proof: See [3]. The proof is based in the known fact that every orbifold is isomorphic to the space of leaves of a foliation with compact leaves and finite holonomy groups [15]. □

COROLLARY 2.3 Let \( H: U \times I \to X \) be an orbifold homotopy of the inclusion. If \( y = H(x, t) \), then \( G_x \) is a subgroup of \( G_y \). That is, an orbifold homotopy preserves the isotropy.

Proof: Since \( \tilde{H}_{(x, t)}: G_x \to G_y \) is injective, we have that \( G_x = \text{im}\tilde{H}_{(x, t)} \) which is a subgroup of \( G_y \). □

3 Stratification

Let \( S_H = \{ x \in X \mid G_x = H \} \). For each \( x \in X \), let \( S_x \) be the connected component of \( S_{G_x} \) containing \( x \). We define a stratification \( S \) of \( X \) as \( S = \{ S_x \}_{x \in X} \). That is, the strata of \( S \) are the connected components of the sets with the same isotropy. Since the regular set is connected, we have that for all the regular points \( x \in X - \Sigma X \) the stratum \( S_x = S_1 = X - \Sigma X \) coincides with the regular set.

PROPOSITION 3.1 [19] Let \( G \) be a compact Lie group acting on a compact manifold \( M \). Then

1. the action has finitely many conjugacy classes of isotropy groups.

2. \( S_{(H)} = \{ z \in M \mid (G_z) = (H) \} \) is a submanifold of \( M \) which may have components of different dimension.
We can always consider all the local isotropy groups as subgroups of a Lie group $G$. Now we introduce a filtration $\mathcal{H}$ of the set $\mathcal{H}_0$ of all the isotropy groups of the action of $G$:

$$\mathcal{H} = \mathcal{H}_0 \supset \mathcal{H}_1 \supset \cdots \supset \mathcal{H}_r = \{G\}$$

where $\mathcal{H}_{i+1} = \{H \leq G| \exists H_i \in \mathcal{H}_i \text{ such that } H \varsupset H_i\}$ for each $0 \leq i \leq r - 1$.

Now, we introduce an orbifold filtration of $X$:

$$X = X^0 \supset X^1 \supset \cdots \supset X^r$$

where

$$X^{i+1} = \{x \in X^i | G_x \in \mathcal{H}_{i+1}\}$$

Alternately, we can see each term of the filtration as $X^i = \bigcup_{H \in \mathcal{H}_i} S_H$.

From Proposition 3.1 follows that the length of the filtration is finite. Note that each $X^i$ is a union of orbifolds and may have several connected components.

We will prove now that an orbifold homotopy of the inclusion preserves the orbifold filtration.

**PROPOSITION 3.2** Let $U$ be an open set of $X$ and $H: U \times I \to X$ be an orbifold homotopy of the inclusion. If $\{X^i | i = 1, \ldots, r\}$ is the orbifold filtration, then $H_t(X^i \cap U) \subset X^j$ for all $t \in I$.

**Proof:** If $x \in X^i$ then $G_x \in \mathcal{H}_i$ since $x$ is in the connected component of $S_{G_x}$ containing $x$. Therefore, there is a subgroup $H \in \mathcal{H}_{i-1}$ such that $G_x \varsupset H$. By Corollary 2.3, $G_x$ is a subgroup of $G_{H(x,t)}$ then $H \varsupset G_x \subset G_{H(x,t)}$ with $H \in \mathcal{H}_{i-1}$ and $H(x,t) \in X^i$. $\square$

Now we give a lower bound for the orbifold category in terms of the category of each term of the filtration.

**PROPOSITION 3.3** $\text{cat}_{\text{orb}} X^i \leq \text{cat}_{\text{orb}} X$ for $i = 1, \ldots, r$

**Proof:** Let $U$ be an orbifold categorical open set for $X$ and $H: U \times I \to X$ be an orbifold homotopy of the inclusion with $H_1(U) = x_0$. By Proposition 3.2 we have that the restriction of $H$ to $X^i$ has image in the same term of the filtration:

$$H: (X^i \cap U) \times I \to X^i$$

then $X^i \cap U$ is an orbifold categorical open set for $X^i$ and $\text{cat}_{\text{orb}} X^i \leq \text{cat}_{\text{orb}} X$.

Even if $X^i$ is not an orbifold, we can extend the definition of $\text{cat}_{\text{orb}} X^i$ in the obvious way: the sum of the orbifold categories of its connected components. $\square$

Then we have that the orbifold category is bounded below by the ordinary category of $X$ and by the orbifold category of the exceptional set:
\[
\max\{\text{cat}_X, \text{cat}_{\text{orb}} \Sigma X\} \leq \text{cat}_{\text{orb}} X
\]

In the ordinary Lusternik-Schnirelmann theory we have that the dimension of a connected topological space is an upper bound for its category [13]:

\[
\text{cat}_X \leq \dim X + 1
\]

This upper bound is not true in general for the orbifold category since we can produce examples of orbifolds of a fixed dimension and arbitrary large orbifold category. For instance, consider \( G = \mathbb{Z}_2 \) acting by rotation on a genus \( g \) surface. The quotient orbifold is topologically an sphere and the singular set consists of \( 2g + 2 \) points. Therefore \( \text{cat}_{\text{orb}} \Sigma X = 2g + 2 \) and \( \text{cat}_{\text{orb}} X \geq 2g + 2 \). We can construct a categorical covering by \( 2g + 2 \) open sets, then \( \text{cat}_{\text{orb}} X = 2g + 2 \) while the dimension of the orbifold is 2.

We give the following version of the dimensional upper bound for the orbifold category:

**PROPOSITION 3.4** If \( \Sigma X \neq \emptyset \), then \( \text{cat}_{\text{orb}} X \leq \text{cat}_{\text{orb}} \Sigma X + \dim X \).

**Proof:** Let \( \{V_1, \cdots, V_m\} \) be an orbifold categorical covering of \( \Sigma X \). For each \( V_i \subset \Sigma X \), there exists an open subset \( U_i \) of \( X \) such that \( V_i \subset U_i \) and \( U_i \) is orbifold categorical [5]. Then \( \{U_1, \cdots, U_m\} \) is an orbifold categorical covering of \( \Sigma X \) by open sets of \( X \).

If \( n = \dim X \), the regular set of the orbifold, \( X - \Sigma X \), is an open manifold of dimension \( n \). We will show that \( X - \Sigma X \) is deformable into a \( (n - 1) \)-dimensional simplicial complex and then, \( \text{cat}_{\text{orb}} (X - \Sigma X) \leq n \).

Take a small triangulation of \( X \) such that the intersection of \( \Sigma X \) with each top dimensional simplex is either contractible or empty [2, 11]. For each top dimensional simplex with empty intersection choose a point \( p \) in its interior, let \( P \) be the set of these points. Choose a path \( \gamma_p \) in \( X \) for each \( p \in P \), which never crosses itself, joining this point \( p \) with a point in \( \Sigma X \) such that \( \gamma_p(I) \cap \gamma_q(I) \subset \Sigma X \) for all \( q \in P, q \neq p \). Let

\[
\Gamma = \bigcup_{p \in P} \gamma_p(I)
\]

So, \( X - \Sigma X \) has the same type of homotopy than \( X - \Sigma X \setminus \Gamma \).

The open subset \( X - \Sigma X \setminus \Gamma \) is deformable into its \( (n - 1) \)-simplicial structure \( T \), so its ordinary category is bounded by the ordinary category of \( T \). Therefore, by the classical dimensional bound for the ordinary category [12], we have \( \text{cat}(X - \Sigma X \setminus \Gamma) \leq n \). Then \( \text{cat}(X - \Sigma X) \leq n \). Since \( X - \Sigma X \) is a manifold, we have that \( \text{cat}(X - \Sigma X) = \text{cat}_{\text{orb}}(X - \Sigma X) \) and we can choose \( \{U_{m+1}, \cdots, U_{m+n}\} \) an orbifold categorical covering of \( X - \Sigma X \) and \( \{U_1, \cdots, U_m, U_{m+1}, \cdots, U_{m+n}\} \) is an orbifold categorical covering of \( X \).

The following example shows that these estimates are optimal.

**EXAMPLE 3.5** Consider the action of \( G = \mathbb{Z}_3 \) on the projective plane \( \mathbb{R}P^2 \) given by the rotation by \( 2\pi/3 \) on the covering 2-sphere \( S^2 \).
The rotation has 2 fixed points on $S^2$, denoted by $\{\pm a\}$. As $G$ has odd order, the quotient action on $\mathbb{RP}^2$ has a unique fixed-point, denoted by $[a]$. Then $\Sigma X = [a]$ and $\text{cat}_{\text{orb}} \Sigma X = 1$. The orbifold $X$ is topologically $\mathbb{RP}^2$ and the ordinary category of $X$ is $\text{cat} X = 3$. Thus,

$$3 = \text{cat} X \leq \text{cat}_{\text{orb}} X \leq \text{cat}_{\text{orb}} \Sigma X + \dim X = 1 + 2 = 3$$

References


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