A QUILLEN MODEL STRUCTURE FOR ORBIFOLDS

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Abstract. A Quillen model structure on the category of orbifold groupoids is constructed. The fibrant objects for this category are the stacks groupoids and the homotopy category is the category of orbifolds.

1. Introduction

In this paper we show that there is a Quillen model structure on the category of orbifold groupoids, in which the weak equivalences are the essential equivalences, the cofibrations are all maps, and the fibrations have the right lifting property with respect to the essential equivalences. The fibrant objects for this model are the stack groupoids, as defined by Grothendieck and Giraud [18].

In sections 2 and 3 we introduce the notion of orbifolds. First in the classical framework of charts and atlases originated by Satake [47] and then we introduce the current approach to orbifolds as groupoids as first proposed by Moerdijk and Pronk [40]. Section 4 presents some equivalent ways to obtain the category of orbifolds. These different approaches will shed light on the Quillen model structure constructed in section 5. The fibrant objects of this model are the stack groupoids, defined in section 6, where a basic background on stacks is also provided. In section 7, we describe the homotopy category of our model as the category of orbifolds.

The approach to constructing the category of orbifolds as a quotient of the bicategory of orbifolds suggests that this model can be extended to be a bicategory of models. We conjecture that there is a model structure in the bicategory of orbifold groupoids whose homotopy bicategory is the bicategory of orbifolds.

2. Classical orbifolds

The notion of orbifold was first introduced by Satake in [47]. His original definition is given in terms of charts, generalizing the notion of manifold structure.

An orbifold chart \((V, G, \pi)\) is given by an open subset \(V \subset \mathbb{R}^n\), a finite group \(G\) acting smoothly on \(V\) and a map \(\pi: V \to X\) inducing a homeomorphism of \(V/G\) onto an open subset \(U \subset X\). An orbifold atlas is a collection \(\mathcal{U}\) of charts covering \(X\) such that for any two charts \((V', G', \pi')\)
and \((V'', G'', \pi'')\) with \(x \in \pi'(V') \cap \pi''(V'')\) there exists a chart \((V, G, \pi)\) in the collection and a pair of injective morphisms from \((V, G, \pi)\) to \((V', G', \pi')\) and \((V'', G'', \pi'')\) respectively so that \(x \in \pi(V)\).

An orbifold atlas \(U\) is a refinement of the orbifold atlas \(U'\) if for every chart in \((V, G, \pi) \in U\) there exists an embedding \(\lambda : (V, G, \pi) \to (V', G', \pi')\) into some chart \((V', G', \pi') \in U'\) with \(\pi' \lambda = \pi\). Two orbifold atlases are said to be equivalent if they have a common refinement.

**Definition 2.1.** A Satake orbifold \(X\) is a Hausdorff space \(X\) equipped with an equivalence class \([U]\) of orbifold atlases.

Satake approach works only for effective orbifolds. We will present in the next section, a more modern approach to orbifolds as groupoids which generalizes the notion of Satake orbifold.

### 3. Orbifolds as Groupoids

A topological groupoid \(\mathcal{G}\) is a groupoid object in the category \(\text{Top}\) of topological spaces and continuous maps. Our notation for groupoids is that \(\mathcal{G}_0\) is the space of objects and \(\mathcal{G}_1\) is the space of arrows, with source and target maps \(s, t : \mathcal{G}_1 \to \mathcal{G}_0\), multiplication \(m : \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \to \mathcal{G}_1\), inversion \(i : \mathcal{G}_1 \to \mathcal{G}_1\), and object inclusion \(u : \mathcal{G}_0 \hookrightarrow \mathcal{G}_1\).

The set of arrows from \(x\) to \(y\) is denoted \(\mathcal{G}(x, y) = \{g \in \mathcal{G}_1 | s(g) = x \text{ and } t(g) = y\}\). The set of arrows from \(x\) to itself, \(\mathcal{G}(x, x)\), is a group called the isotropy group of \(\mathcal{G}\) at \(x\) and denoted \(\mathcal{G}_x\).

A strict morphism \(\phi : \mathcal{K} \to \mathcal{G}\) of groupoids is a functor given by two continuous maps \(\phi : \mathcal{K}_1 \to \mathcal{G}_1\) and \(\phi : \mathcal{K}_0 \to \mathcal{G}_0\) that together commute with all the structure maps of the groupoids \(\mathcal{K}\) and \(\mathcal{G}\).

A natural transformation \(T\) between two morphisms \(\phi, \psi : \mathcal{K} \to \mathcal{G}\) is a continuous map \(T : \mathcal{K}_0 \to \mathcal{G}_1\) with \(T(x) : \phi(x) \to \psi(x)\) such that for any arrow \(h : x \to y\) in \(\mathcal{K}_1\), the identity \(\psi(h)T(x) = T(y)\phi(h)\) holds. We write \(\phi \sim_T \psi\).

Topological groupoids, strict morphisms and natural transformations form a 2-category that we denote \(\text{TopGpd}\). (We use the same notation also for the underlying 1-category of topological groupoids and strict morphisms.)

A topological groupoid \(\mathcal{G}\) is proper if the diagonal \((s, t) : \mathcal{G}_1 \to \mathcal{G}_0 \times \mathcal{G}_0\) is a proper map. Note that in this case the isotropy groups are compact.

A topological groupoid \(\mathcal{G}\) is a foliation groupoid if each isotropy group is discrete.

**Definition 3.1.** An orbifold groupoid is a proper foliation groupoid.

Note that in this case the isotropy groups are finite.

Every Satake orbifold \(X\) has associated an orbifold groupoid \(\mathcal{G}_X\) and reciprocally, every smooth effective orbifold groupoid \(\mathcal{G}\) has an orbit space \(X = |\mathcal{G}|\) equipped with a Satake orbifold structure [1].
In this way, we can think of an orbifold groupoid as an atlas for the orbifold. We will define next a notion of equivalence for groupoids that will imitate the notion of equivalence of atlases.

A strict morphism \( \epsilon : K \to G \) of topological groupoids is an essential equivalence if

(i) \( \epsilon \) is essentially surjective in the sense that

\[
s\pi_1 : G_1 \times^t_{G_0} K_0 \to G_0
\]

is an open surjection where \( G_1 \times^t_{G_0} K_0 \) is the pullback along the target \( t : G_1 \to G_0 \);

(ii) \( \epsilon \) is fully faithful in the sense that \( K_1 \) is the following pullback of topological spaces:

\[
\begin{array}{c}
K_1 \\
\downarrow (s,t) \downarrow \\
K_0 \times K_0 \xrightarrow{\epsilon \times \epsilon} G_0 \times G_0
\end{array}
\]

An essential equivalence \( \epsilon : K \to G \) does not generally have an inverse functor \( \delta : G \to K \) such that \( \epsilon \circ \delta \sim T \text{id}_G \) and \( \delta \circ \epsilon \sim T' \text{id}_K \) in TopGpd. The functor \( \delta \) exists by the axiom of choice but in general it is not continuous.

**Proposition 3.2.** Proper groupoids and foliation groupoids are invariant under essential equivalence.

**Definition 3.3.** Let \( \psi : K \to G \) and \( \phi : L \to G \) be strict morphisms. The weak pullback \( P = K \times^G L \) is the topological groupoid whose space of objects is \( P_0 = K_0 \times^G L_0 \) and space of arrows is \( P_1 = K_1 \times^G L_1 \). Source and target maps are given by \( s(k, g, l) = (s(k), \psi(k)^{-1} g \phi(l), s(l)) \) and \( t(k, g, l) = (t(k), g, t(l)) \). There is a square of morphisms

\[
\begin{array}{c}
K \times G L \\
\downarrow \pi_2 \downarrow \psi \\
L \xrightarrow{\phi} G
\end{array}
\]

which commutes up to a natural transformation, and is universal with this property.

**Definition 3.4.** The orbifold groupoids \( K \) and \( G \) are Morita equivalent if there exists an orbifold groupoid \( L \) and a span

\[
K \xleftarrow{\sigma} L \xrightarrow{\epsilon} G
\]

where \( \epsilon \) and \( \sigma \) are essential equivalences. We write \( G \sim_M K \).

The proof that a Morita equivalence is an equivalence relation is based in the weak pullback defined above.
A *generalized map* from $\mathcal{K}$ to $\mathcal{G}$ is a span $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ such that $\epsilon$ is an essential equivalence. Two generalized maps $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ and $\mathcal{K} \xleftarrow{\epsilon'} \mathcal{J}' \xrightarrow{\phi'} \mathcal{G}$ are *equivalent* if there exists a diagram

\[
\begin{array}{ccc}
\mathcal{J} & \xrightarrow{\phi} & \mathcal{G} \\
\downarrow{\epsilon} & & \uparrow{\phi'} \\
\mathcal{K} & \xleftarrow{u} & \mathcal{L} \\
\downarrow{v} & & \uparrow{\phi'} \\
\mathcal{J}' & \xrightarrow{\epsilon'} & \mathcal{G}
\end{array}
\]

which is commutative up to natural transformations and where $\mathcal{L}$ is an orbifold groupoid, and $u$ and $v$ are essential equivalences.

An *orbifold structure* $\mathcal{O}$ on a Hausdorff space $X$ is a pair $(\mathcal{G}, f)$ where $\mathcal{G}$ is an orbifold groupoid and $f : |\mathcal{G}| \to X$ is a homeomorphism. Two orbifold structures $(\mathcal{G}, f)$ and $(\mathcal{K}, h)$ are *equivalent* if $\mathcal{G} \sim_M \mathcal{K}$ and $h \bar{\sigma} = f \bar{\epsilon}$, where $\bar{\sigma}$ and $\bar{\epsilon}$ are the morphisms induced by the Morita equivalence in the orbit spaces $|\mathcal{K}| \xleftarrow{\bar{\sigma}} |\mathcal{L}| \xrightarrow{\bar{\epsilon}} |\mathcal{G}|$.

**Definition 3.5.** An *orbifold* $\mathcal{X}$ is a Hausdorff space $X$ equipped with an equivalence class $[\mathcal{O}]$ of orbifold structures. We write $\mathcal{X} = (X, [\mathcal{O}])$.

Next we define a notion of orbifold map between orbifolds.

**Definition 3.6.** An *orbifold map* $\mathcal{Y} \to \mathcal{X}$ is a map $Y \to X$ between the underlying topological spaces and an equivalence class of generalized maps $[\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}]$, where $\mathcal{X} = (X, [(\mathcal{G}, f)])$ and $\mathcal{Y} = (Y, [(\mathcal{K}, h)])$.

The category of orbifolds has orbifolds $\mathcal{X}$ as objects and orbifold maps $\mathcal{Y} \to \mathcal{X}$ as arrows. We denote this category *Orbifolds*.

In the next section, we will discuss different approaches to the description of the category of orbifolds.

### 4. The Category of Orbifolds

We denote $\text{OrbGpd}$ the sub 2-category of orbifold groupoids (and the underlying category). We denote $\text{OrbGrd}$ the category of orbifold groupoids and isomorphism classes of strict morphisms. (Recall that two morphisms are isomorphic is there is a natural transformation between them.)

Consider the class of arrows $E$ given by the essential equivalences in the 2-category $\text{OrbGpd}$. It was proven by Pronk in [44] that $E$ satisfies the conditions to admit a bicalculus of fractions. The bicategory of fractions $\text{OrbGpd}(E^{-1})$ obtained by formally inverting the essential equivalences is what we call the *bicategory of orbifolds* and we denote $\text{Orb}$.

The explicit description of the bicategory $\text{Orb}$ is as follows:

- Objects are orbifold groupoids $\mathcal{G}$.
• A 1-morphism from \( \mathcal{K} \) to \( \mathcal{G} \) is a generalized map

\[
\mathcal{K} \overset{\epsilon}{\leftarrow} \mathcal{J} \overset{\phi}{\rightarrow} \mathcal{G}
\]

such that \( \epsilon \) is an essential equivalence.

• A 2-morphism from \( \mathcal{K} \overset{\epsilon}{\leftarrow} \mathcal{J} \overset{\phi}{\rightarrow} \mathcal{G} \) to \( \mathcal{K} \overset{\epsilon'}{\leftarrow} \mathcal{J}' \overset{\phi'}{\rightarrow} \mathcal{G} \) is given by a class of diagrams:

\[
\begin{array}{c}
\mathcal{K} \\
\downarrow \phi \\
\mathcal{G} \\
\downarrow \phi' \\
\mathcal{L} \\
\downarrow \psi \\
\mathcal{J}' \\
\end{array}
\]

where \( \mathcal{L} \) is an orbifold groupoid, and \( u \) and \( v \) are essential equivalences.

The horizontal composition of generalized maps \( \mathcal{K} \overset{\epsilon}{\leftarrow} \mathcal{J} \overset{\phi}{\rightarrow} \mathcal{G} \) and \( \mathcal{G} \overset{\zeta}{\leftarrow} \mathcal{J}' \overset{\psi}{\rightarrow} \mathcal{L} \) is given by the diagram

\[
\begin{array}{c}
\mathcal{J}' \times_{\mathcal{G}} \mathcal{J} \\
\downarrow \phi \\
\mathcal{J} \\
\downarrow \psi \\
\mathcal{L} \\
\end{array}
\]

where \( \mathcal{J}' \times_{\mathcal{G}} \mathcal{J} \) is the weak pullback of groupoids.

**Remark 4.1.** There is no underlying category here since this composition of arrows is not associative on the nose.

Let \( \textbf{Orb} \) be the 1-category obtained from \( \text{Orb} \) by identifying isomorphic morphisms, \( \textbf{Orb} = \overline{\text{OrbGpd}}(E^{-1}) \). Recall that two 1-morphisms in a bicategory are isomorphic if there are 2-morphisms \( a \) and \( b \)

\[
\begin{array}{c}
\mathcal{K} \\
\downarrow a \\
\mathcal{G} \\
\end{array}
\]

such that \( a \circ b = \text{id} \) and \( b \circ a = \text{id} \).

Since all the 2-morphisms in \( \text{Orb} \) are isomorphisms, we have that the objects of \( \text{Orb} \) are orbifold groupoids and the morphisms are equivalence classes of generalized maps where \( (\mathcal{K} \overset{\epsilon}{\leftarrow} \mathcal{J} \overset{\phi}{\rightarrow} \mathcal{G}) \sim (\mathcal{K} \overset{\epsilon'}{\leftarrow} \mathcal{J}' \overset{\phi'}{\rightarrow} \mathcal{G}) \) if there
exists a diagram:

\[
\begin{array}{c}
\mathcal{K} \\
\downarrow^\sim \mathcal{L} \\
\downarrow^v \mathcal{J} \\
\downarrow^\phi \mathcal{J}' \\
\downarrow^u \mathcal{G} \\
\end{array}
\]

where \( \mathcal{L} \) is an orbifold groupoid, and \( u \) and \( v \) are essential equivalences. We denote by \(|K \xleftarrow{\epsilon} J \xrightarrow{\phi} G|\) the morphisms in \( \text{Orb} \).

Alternately, we can obtain \( \text{Orb} \) also as a localization of the category \( \text{OrbGrd} \) by the class of arrows \( \mathbf{E} \) given by the isomorphism classes of essential equivalences. It was proven by Moerdijk in [37] that \( \mathbf{E} \) satisfy the Gabriel and Zisman conditions to admit a calculus of fractions. The category of fractions \( \text{OrbGrd}[\mathbf{E}^{-1}] \) obtained by formally inverting the isomorphism classes of essential equivalences is equivalent to the category of orbifolds.

**Remark 4.2.** The category of orbifolds can therefore be described in terms of orbifold structures, as a quotient of the bicategory of orbifolds and as a localization of a category of orbifold groupoids:

\[
\text{Orbifolds} \sim \text{Orb} = \overline{\text{OrbGpd}}(\mathbf{E}^{-1}) \sim \text{OrbGrd}[\mathbf{E}^{-1}]
\]

5. A model structure for \( \text{OrbGrd} \)

The goal of this section is to prove that there is a Quillen model structure on \( \text{OrbGrd} \) in which the weak equivalences are the isomorphism classes of essential equivalences and whose homotopy category is \( \text{Orbifolds} \), the category of orbifolds as described above.

The category \( \text{OrbGrd} \) has products and coproducts but it does not have pullbacks and pushouts. Therefore does not satisfy the requirement of completeness and cocompleteness to be a model category in the sense of Quillen. We will prove that it does satisfy the remaining Quillen axioms, so there is a *model structure* in \( \text{OrbGrd} \) (as defined by Hovey [24]). Moreover, since \( \text{OrbGrd} \) has finite products and coproducts and has a model structure, we can still construct the homotopy category as showed in [16].

All objects for this model structure will be cofibrant and we will prove that the fibrant objects are exactly the *stack groupoids* as defined by Joyal in [27].

5.1. **Model categories.** A *model structure* on a category consists of three interrelated classes of maps called the weak equivalences, the cofibrations, and the fibrations, satisfying certain axioms. The weak equivalences determine the homotopy category since the axioms are designed to lead to a well-behaved homotopy category that is obtained by inverting the weak equivalences. The references for this section are [46, 35, 24, 23].
Definition 5.1. A model structure on a category $C$ consist of three classes of maps – weak equivalences, fibrations, and cofibrations – subject to the following axioms. Fibrations and cofibrations that are also weak equivalences are called, respectively, trivial fibrations and trivial cofibrations.

M1. (Retract Axiom) The three distinguished classes of maps are closed under retracts.
M2. (2 out of 3) Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ so that any two of $f$, $g$, or $gf$ is a weak equivalence, then so is the third.
M3. (Lifting Axiom) Every lifting problem
$$
\begin{array}{c}
A \\
\downarrow j \quad \downarrow q
\end{array}
\xrightarrow{j} \quad \xrightarrow{q}
\begin{array}{c}
X \\
\downarrow q
\end{array}
\xleftarrow{B} \xrightarrow{Y}
$$
where $j$ is a cofibration and $q$ is a fibration has a solution so that both diagrams commute if one of $j$ or $q$ is a weak equivalence.
M4. (Factorization) Any $f : X \rightarrow Y$ can be factored two ways:
(i) $X \xrightarrow{i} Z \xrightarrow{q} Y$, where $i$ is a cofibration and $q$ is trivial fibration.
(ii) $X \xrightarrow{j} Z \xrightarrow{p} Y$, where $j$ is trivial cofibration and $p$ is a fibration.

Definition 5.2. A model category is a category $C$ closed under finite limits and colimits together with a model structure on $C$.

We state next our main result:

Theorem 5.3. There is a model structure in the category $\text{OrbGrd}$ where the weak equivalences are the isomorphism classes of essential equivalences, the cofibrations are all maps and the fibrations are the morphisms in $\text{OrbGrd}$ with the right lifting property with respect to the weak equivalences.

We present first some preliminary results.

Proposition 5.4. [44] Let $\epsilon : K \rightarrow G$ be an essential equivalence and $\phi, \varphi : L \rightarrow K$ be functors. If $\epsilon \phi \sim_T \epsilon \varphi$ then $\phi \sim_T \varphi$.

Proposition 5.5. [45] If two out of $\{\phi, \psi, \phi \circ \psi\}$ are essential equivalences, then so is the third.

Proposition 5.6. If $\epsilon \sim_T \varphi$ and $\epsilon$ is an essential equivalence, then $\varphi$ is an essential equivalence.

Our next goal is to show a necessary and sufficient condition for an essential equivalence to be an equivalence. This condition will be determinant for the characterization of fibrant objects of our model.

Lemma 5.7. Let $\epsilon : K \rightarrow G$ be an essential equivalence. If $\epsilon$ has a section or a retraction up to isomorphism, then $\epsilon$ is an equivalence.
Proof. If there exists a section \( \sigma \) such that \( \epsilon \sigma \sim \text{id}_G \), we have that \( \epsilon \sigma \epsilon \sim \epsilon \).

Then \( \sigma \epsilon \sim \text{id}_K \) by Proposition 5.4. If \( \epsilon \) has a retraction \( \rho \) such that \( \rho \epsilon \sim \text{id}_K \), then \( \rho \epsilon \) is an essential equivalence by Proposition 5.6. And then \( \rho \) is an essential equivalence by Proposition 5.5. We have that \( \rho \epsilon \rho \sim \rho \) and the result follows again by Proposition 5.4. \( \square \)

**Proposition 5.8.** Let \( \epsilon : K \to G \) be an essential equivalence and

\[ s\pi_1 : G_1 \times_{G_0} K_0 \to G_0 \]

the open surjection given by the essential surjectivity of \( \epsilon \). Then \( \epsilon \) is an equivalence if and only if \( s\pi_1 \) has a global section.

Proof. If \( \epsilon \) is an equivalence, then there exists a morphism \( \eta \) such that \( \epsilon \eta \sim T \text{id} \), where \( T : G_0 \to G_1 \) is a natural transformation. Define \( \sigma : G_0 \to G_1 \times_{G_0} K_0 \) as \( \sigma(y) = (T(y), \eta(y)) \). Then \( s\pi_1 \sigma(y) = s\pi_1(T(y), \eta(y)) = sT(y) = y \) since \( T(y) \) is an arrow from \( y \) to \( \epsilon(\eta(y)) \).

Reciprocally, if \( \sigma : G_0 \to G_1 \times_{G_0} K_0 \) is a section of \( s\pi_1 \) we have that \( \sigma(y) = (g, x) \in G_1 \times_{G_0} K_0 \) with \( s(g) = y \) and \( t(g) = \epsilon(x) \). Define \( \eta : G \to K \) on objects as \( \eta(y) = x \) and extend it to arrows using fully faithfulness. Then \( \text{id}_G \sim_T \epsilon \eta \) with \( : G_0 \to G_1 \) given by \( T(y) = g \).

Then \( \eta \) is a section of \( \epsilon \) and by lemma 6.6, we have that \( \epsilon \) is an equivalence. \( \square \)

Recall that \( \text{OrbGrd} \) is the category of orbifold groupoids and isomorphism classes of functors. In other words, an arrow \( \bar{\phi} : K \to G \) in \( \text{OrbGrd} \) is given by \( \bar{\phi} = \{ \phi : K \to G | \exists T \text{ with } \phi \sim_T \phi \} \).

(We drop now the particular natural transformation \( T \) and use the notation \( \phi \sim \phi \).)

**Remark 5.9.** A class \( \bar{\epsilon} \) is a weak equivalence if there exists an essential equivalence \( \sigma \) such that \( \epsilon \sim \sigma \).

In this case, by proposition 5.6, we have that then all functors in the class \( \bar{\epsilon} \) are essential equivalences as well.

**Proposition 5.10.** The class \( \bar{p} : K \to G \) has the right lifting property with respect to the class \( \bar{\phi} : A \to B \) if and only if \( q \) has the right lifting property with respect to \( \varphi \) up to a natural transformation for all \( q \in \bar{p} \) and \( \varphi \in \bar{\phi} \).

**Remark 5.11.** A class \( \bar{p} \) is a fibration if for every commutative square up to a natural transformation

\[
\begin{array}{ccc}
A & \longrightarrow & K \\
\varphi \downarrow & & \gamma \downarrow q \\
B & \longrightarrow & G
\end{array}
\]

where \( q \in \bar{p}, \varphi \in \bar{\phi} \) and \( \varphi \) is an essential equivalence, there is an arrow \( \gamma \) such that both triangles commute up to natural transformations.
Remark 5.12. A class $\phi$ is an isomorphism in $\text{OrbGrd}$ if there exists an equivalence $\varphi$ in $\text{OrbGpd}$ such that $\phi \sim \varphi$. In this case, all the functors in the class $\phi$ are equivalences in $\text{OrbGpd}$.

Proposition 5.13. Trivial fibrations are the isomorphism classes of equivalences (isomorphism in $\text{OrbGrd}$) and trivial cofibrations are the isomorphism classes of essential equivalences (weak equivalences in $\text{OrbGrd}$).

Proof. If $\epsilon$ is a trivial fibration, then $\epsilon$ is an essential equivalence and has the RLP with respect to essential equivalences. Therefore, the following square has a lifting $\gamma$ up to natural transformations:

$$
\begin{array}{ccc}
K & \to & K \\
\downarrow \epsilon & & \downarrow \epsilon \\
G & \to & G \\
\end{array}
$$

Then $\epsilon$ is an equivalence.

Clearly, if $\sigma$ is a trivial cofibration, then $\sigma$ is an essential equivalence. □

Remark 5.14. A class $\phi$ is a retract of $\rho$ if there is commutative diagram up to isomorphism of the following form:

$$
\begin{array}{ccc}
K & \to & L \to K \\
\downarrow \varphi & & \downarrow \varphi \\
G & \to & J \to G \\
\end{array}
$$

where $\varphi \in \phi$, $q \in \rho$ and the horizontal composites are equivalent to identities.

Proof of theorem 5.3. Since cofibrations are all maps and fibrations are defined by a lifting property, they both are closed under retracts. It remains to prove that if $\epsilon$ is a weak equivalence, then a retract of $\epsilon$ is weak equivalence as well. To show this, we will prove that the essential equivalences are closed under retracts up to isomorphism.

M2 follows from propositions 5.5 and 5.6. The only part of M3 that is needed to prove is that there exists a lifting $\gamma$ for every diagram

$$
\begin{array}{ccc}
K & \to & L \\
\downarrow j & & \downarrow q \\
G & \to & J \\
\end{array}
$$

where $qa \sim bj$ and $q$ is an equivalence. In this case, there exists $p: J \to L$ such that $pq \sim \text{id}$ and $qp \sim \text{id}$. Let $\gamma = pb$ and we have established M3.

Given a map $\phi: K \to G$ the first half of M4 is trivial considering the following
factorization as a cofibration followed by a trivial fibration:

\[
\begin{array}{ccc}
\mathcal{K} & \xrightarrow{\phi} & \mathcal{G} \\
\downarrow & & \downarrow \\
\mathcal{G} & \xrightarrow{id} & \mathcal{G}
\end{array}
\]

For the second part of M4 we need to construct a groupoid \( \mathcal{K}^* \) and a factorization

\[
\begin{array}{ccc}
\mathcal{K} & \xrightarrow{\phi} & \mathcal{G} \\
\downarrow & & \downarrow \\
\mathcal{K}^* & \xrightarrow{i} & \mathcal{G} \\
\downarrow & & \downarrow \\
\mathcal{L} & \xrightarrow{p} & \mathcal{L}
\end{array}
\]

such that \( pi \sim \phi \), \( i \) is an essential equivalence and \( p \) has the RLP with respect to essential equivalences. Let \( \mathcal{K}^* \) be the weak pullback of Haefliger’s classifying groupoid \( H \mathcal{G} \) (as defined in appendix) and \( \mathcal{L} \):

\[
\mathcal{K}^* = H \mathcal{G} \times_{\mathcal{L}} \mathcal{L} \rightarrow H \mathcal{G}
\]

\[
\begin{array}{ccc}
\mathcal{L} & \xrightarrow{} & \mathcal{L}
\end{array}
\]

\( \square \)

**Remark 5.15.** We can always find a factorization as in M4 where the cofibration is injective on objects. Consider \( \mathcal{G}^* \) given by

\[
(K_0 \sqcup G_0) \times_{G_0}^I G_1 \times_{G_0}^s (K_0 \sqcup G_0) \rightarrow K_0 \sqcup G_0
\]

where \( s(x, g, y) = x \) and \( t(x, g, y) = y \). The cofibration \( i : \mathcal{K} \rightarrow \mathcal{G}^* \) is defined on objects by \( i(x) = i_{K_0}(x) \) where \( i_{K_0} : K_0 \rightarrow K_0 \sqcup G_0 \) is the inclusion; and on arrows by \( \phi(k) = (i_{K_0} s(k), \phi(k), i_{K_0} t(k)) \). Let \( p : \mathcal{G}^* \rightarrow \mathcal{G} \) be the functor defined on objects by \( p(x) = \phi(x) \) if \( x \in K_0 \) and \( p(x) = x \) if \( x \in G_0 \); and on arrows by \( p(x, g, y) = g \). We have that \( p \) is an equivalence and \( i \) is injective on objects.

### 5.2. Fibrant and cofibrant objects.

The initial object in \( \text{OrbGrd} \) is \( \emptyset \) and the final object is the trivial groupoid \( * \) with one object and one arrow. Recall that an object \( \mathcal{G} \) is **cofibrant** if the morphism \( \emptyset \rightarrow \mathcal{G} \) is a cofibration and it is **fibrant** if \( \mathcal{G} \rightarrow * \) is a fibration.

All groupoids in \( \text{OrbGrd} \) are cofibrant. We want to characterize the fibrant objects of our model.

**Proposition 5.16.** A groupoid \( \mathcal{G} \) is fibrant iff for all diagrams

\[
\begin{array}{ccc}
A & \xrightarrow{\phi} & \mathcal{G} \\
\downarrow \varepsilon & & \downarrow \gamma \\
B & \xrightarrow{} &
\end{array}
\]
where $\varepsilon$ is an essential equivalence, there exists a map $\gamma$ making the diagram commutative up to a natural transformation.

Observe that this condition amounts to say that every generalized map whose codomain is $G$ is equivalent to a strict map:

$$B \xleftarrow{\varepsilon} A \xrightarrow{\phi} G \sim B \xrightarrow{\gamma} G$$

**Example 5.17.** A nonfibrant groupoid $G$. Consider the holonomy groupoid associated to the Seifert fibration on the Möbius band, $B = \text{Hol}(M, F_S)$, and the reduced holonomy groupoid $G = \text{Hol}_I(M, F_S)$ to an interval $I$ transversal to the fibers. The space of objects $B_0$ is the Möbius band $M$ and the space of arrows is $B_1 = S^1 \times M$. Similarly, $G_0 = I$ and $G_1$ is given by two disjoint copies of the interval $I$. The inclusion functor $i_G : G \hookrightarrow B$ is an essential equivalence [11]. We have the following diagram:

$$\begin{array}{ccc}
G & \longrightarrow & G \\
\downarrow{i_G} & & \\
B & & \\
\end{array}$$

Suppose that there exists $\gamma : B \to G$ such that $\gamma i_G \sim \text{id}$. Then $i_G$ is an equivalence by lemma 6.6 and by proposition 5.8 this means that there exists a section of the map $s\pi_1 : S^1 \times M \times I \to M$. But $s\pi_1$ is the double covering of the Möbius band by the cylinder $S^1 \times I$ which does not have a section. This is a contradiction, so there is no map $\gamma : B \to G$ such that $\gamma i_G \sim \text{id}$. Therefore $G$ is not fibrant.

**Example 5.18.** Fibrant groupoids. Let $G$ be the pair groupoid, with $G_0 = X$ and $G_1 = X \times X$. Then $G \to \ast$ is a fibration since it is an equivalence. In the other extreme, we have that the unit groupoid $K$, with $K_0 = X$ and $K_1 = X$ is also a fibrant object.

Our next goal is to prove that the fibrant objects of our model structure are the stack groupoids. First we will introduce the basics on topological stacks.

### 6. Stacks

We will describe here stacks as categories fibered on groupoids. The basic references for this section are [6] and [42].

#### 6.1. Categories fibered in groupoids

A category fibered in groupoids over $\textbf{Top}$ is a category $\mathcal{C}$ together with a functor $\pi : \mathcal{C} \to \textbf{Top}$ satisfying the following properties:

1. Given $f : X \to Y$ and $\bar{Y} \in \pi^{-1}(Y)$ there exist a morphism $F : \bar{X} \to \bar{Y}$ such that $\pi F = f$ where $\bar{X} \in \pi^{-1}(X)$.
2. If $F' : X' \to Y$ is another such morphism with $\bar{X}' \in \pi^{-1}(X)$, then there exists an isomorphism $\alpha$ such that $\alpha F = F'$. 


The category of all objects in $C_0$ lying over a topological space $X$ and morphisms in $C_1$ lying over $\text{id}_X$ is called the fiber of $C$ over $X$ and denoted $C_X$. These fibers are groupoids.

A 1-morphism of categories fibered in groupoids over $\text{Top}$ is a functor $F : C \to C'$ such that $\pi'F = \pi$.

A 2-morphism $u$ from $F$ to $G$ is a natural transformation such that $\pi'(u(x)) = \text{id}_{\pi(x)}$ for all $x \in C_0$.

Categories fibered in groupoids with these notions of 1- and 2-morphisms form a 2-category $[19]$. We will denote this 2-category by $\text{CFG}$.

6.2. **Stacks.** A category fibered in groupoids $\pi : C \to \text{Top}$ is a stack if it satisfies the following axioms:

1. for any $X \in \text{Top}$, any two objects $x, y \in C_0$ lying over $X$, and any two isomorphisms $\phi, \psi : x \to y$ over $X$, such $\phi|U_i = \psi|U_i$, for all $U_i$ in a covering family $U_i \to X$, we have that $\phi = \psi$;
2. for any $X \in \text{Top}$, any two objects $x, y \in C_0$ lying over $X$, a covering family $U_i \to X$ and, for every $i$, an isomorphism $\phi_i : x|U_i \to y|U_i$, such that $\phi_i|U_{ij} = \phi_j|U_{ij}$, for all $i, j$, there exists an isomorphism $\phi : x \to y$, such that $\phi|U_i = \phi_i$, for all $i$;
3. for every $X$, every covering family $\{U_i\}$ of $X$, every family $\{x_i\}$ of objects $x_i$ in the fiber $C_{U_i}$ and every family of morphisms $\{\phi_{ij}\}$, $\phi_{ij} : x_i|U_{ij} \to x_j|U_{ij}$, satisfying the cocycle condition $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$, there exists an object $x$ over $X$, together with isomorphisms $\phi_i : x|U_i \to x_i$ such that $\phi_{ij} \circ \phi_i = \phi_j$ (over $U_{ij}$).

The 2-category of stacks $\text{St}$ is the full sub 2-category of $\text{CFG}$ whose objects are stacks.

6.3. **A category fibered in groupoids associated to a topological groupoid.** Given a topological groupoid $\mathcal{G}$, we can construct a category fibered in groupoids $\pi : C \to \text{Top}$ such that each fiber is given by the groupoid

$$(C_X)_0 = \text{Hom}(X, G_0)$$

$$(C_X)_1 = \text{Hom}(X, G_1)$$

and pullback functor is induced by composition. We will denote $[\mathcal{G}]$ the category fibered in groupoids thus constructed. The category fibered in groupoids $[\mathcal{G}]$ is not in general a stack.

We can, however, associate to each category fibered in groupoids $[\mathcal{G}]$ a stack $[\mathcal{G}]$. To do this, we need to introduce first the notion of $\mathcal{G}$-torsor.

6.4. **Torsors.** Let $\mathcal{G}$ be a topological groupoid, $X$ a topological space and $a : E \to G_0$ a map of topological spaces. A $\mathcal{G}$-torsor over $X$ is an open surjection $p : E \to X$ equipped with an action $G_1 \times_{G_0} E \to E$ of $\mathcal{G}$ on the anchor map $a : E \to G_0$ such that

$$G_1 \times_{G_0} E \to E \times_X E$$

is a homeomorphism.
A morphism \((\phi, f)\) from a \(G\)-torsor over \(X\) to a \(G\)-torsor over \(X'\) is given by a commutative diagram

\[
\begin{array}{ccc}
E & \xrightarrow{\phi} & E' \\
\downarrow{\pi} & & \downarrow{\pi'} \\
X & \xrightarrow{f} & X'
\end{array}
\]

where \(\phi\) is \(G\)-invariant. Note that morphisms of \(G\)-torsors over a fixed \(X\) are invertible.

Let \(B\) be the category whose objects are \(G\)-torsos and whose arrows are morphisms of \(G\)-torsors.

6.5. **A stack associated to a topological groupoid.** We can associate to each groupoid \(G\) a stack \([G]\), called the stack completion of \([G]\). It is defined as the category fibered in groupoids \(\pi : B\to \text{Top}\) such that \(\pi(E\to G) = X\) on objects and \(\pi((\phi, f)) = f : X\to X'\) on arrows. Each fiber is given by the groupoid

\[
(BG)_0 = G\text{-torsors over } X \\
(BG)_1 = \text{morphisms of } G\text{-torsors over } X
\]

The category fibered in groupoids \([G]\) thus defined, is a stack.

In the 2-category \(\text{St}\) we can say that two stacks are equal, isomorphic or equivalent. These are three different notions and we will use the symbol \(\sim\) to denote equivalence of stacks. That is, \([G]\sim[G']\) iff there are 1-morphisms of categories fibered in groupoids \(F : [G]\to[G']\) and \(H : [G']\to[G]\) with 2-morphisms \(u : FH \Rightarrow id\) and \(v : HF \Rightarrow id\).

If two groupoids are Morita equivalent, their associated stacks are equivalent as stacks.

**Proposition 6.1.** If \(G\sim_M G'\) then \([G]\sim[G']\).

To each groupoid \(G\) we can associate a category fibered in groupoids \([G]\) and a stack \([G]\). There is a canonical morphism of categories fibered in groupoids \(F : [G]\to[G]\). All groupoids in the same Morita equivalence class will have equivalent associated stacks. However, some of the groupoids in that equivalence class will have an associated category fibered in groupoids \([G]\) that is not a stack, whereas some others in the same class will have an associated \([G']\) that is already a stack. We want to single out these latter groupoids. We will describe the groupoids \(G\) for which \(F : [G]\to[G]\) is an equivalence of categories fibered in groupoids, and therefore \([G]\) is a stack.

**Definition 6.2.** A groupoid \(G\) is a stack groupoid if \([G]\sim[G]\).

We will prove in the next section that these stack groupoids are exactly the fibrant objects of our model.
6.6. **Stack groupoids.** We want to characterize the topological groupoids \( \mathcal{G} \) for which \( \lfloor \mathcal{G} \rfloor \sim [\mathcal{G}] \).

**Proposition 6.3.** [36] Let \( F : \mathcal{C} \to \mathcal{C}' \) be a morphism of categories fibered in groupoids over \( \text{Top} \). Then \( F \) is an equivalence if and only if for each topological space \( X \) the functor \( F_X : \mathcal{C}_X \to \mathcal{C}'_X \) is an equivalence of groupoids.

We will write explicitly the effect of the functor \( F : \lfloor \mathcal{G} \rfloor \to [\mathcal{G}] \) on the fiber over \( X \) for the categories fibered in groupoids \( \lfloor \mathcal{G} \rfloor \) and \( [\mathcal{G}] \). Recall that the fiber \( \lfloor \mathcal{G} \rfloor_X \) is the groupoid:

\[
\text{Hom}(X, G_1) \xrightarrow{\sim} \text{Hom}(X, G_0)
\]

and the fiber \( [\mathcal{G}]_X \) is the groupoid:

Morphisms of \( \mathcal{G} \)-torsors over \( X \) are \( \mathcal{G} \)-torsors over \( X \).

The functor \( F_X : \lfloor \mathcal{G} \rfloor_X \to [\mathcal{G}]_X \) is given by \( F_X( X \xrightarrow{f} G_0 ) = f^*G_1 \to X \) on objects. Where \( f^*G_1 \to X \) is the \( \mathcal{G} \)-torsor given by \( p_2 \) in the following pullback:

\[
f^*G_1 = G_1 \times^f_{G_0} X \xrightarrow{p_1} G_1
\]

\[
p_2 \downarrow \quad \quad \quad \quad \downarrow t
\]

\[
X \xrightarrow{f} G_0
\]

On arrows, \( F_X( X \xrightarrow{h} G_1 ) = f^*G_1 \to g^*G_1 \) where \( sh = f \) and \( th = g \).

Explicitly, \( F_X(h)(k, x) = (h(x)k, x) \) for all \( (k, x) \in f^*G_1 = G_1 \times^h_{G_0} X \).

We have that \( F_X \) is always full and faithful. It is essentially surjective if and only if for every \( \mathcal{G} \)-torsor \( E \to X \), there is a map \( f : X \to G_0 \) such that \( F_X(f) \) and \( E \to X \) are connected by a morphism of \( \mathcal{G} \)-torsors. In other words, if there exists \( f : X \to G_0 \) such that the following diagram commutes:

\[
f^*G_1 \xrightarrow{\sim} E
\]

\[
\downarrow \quad \quad \quad \quad \downarrow
\]

\[
X
\]

**Proposition 6.4.** A topological groupoid \( \mathcal{G} \) is a stack groupoid if and only if for every \( \mathcal{G} \)-torsor \( E \) over \( X \) there is a map \( f : X \to G_0 \) such that \( f^*G_1 \) and \( E \) are isomorphic as \( \mathcal{G} \)-torsors over \( X \).

**Remark 6.5.** This last characterization makes our notion of stack groupoid to coincide with the notion of stack used by Joyal in [27]. Observe that the fibrant objects of his model structure on the category of groupoids in a topos are the strong stacks whereas the fibrant objects in our model are the ordinary stacks, introduced by Grothendieck and Giraud [18].

**Proposition 6.6.** [27] \( \mathcal{G} \) is a stack groupoid if and only if every \( \mathcal{G} \)-torsor has a section.
Proposition 6.7. The fibrant objects of our model structure in OrbGrd are the stacks groupoids.

We will prove this result through a series of lemmas.

Lemma 6.8. Let \( \epsilon : G \to L \) be an essential equivalence. If \( G \) is a stack groupoid then \( \epsilon \) is an equivalence.

Proof. Consider the following pullback of topological spaces:

\[
\begin{array}{c}
L_1 \times_{L_0} G_0 \downarrow \downarrow p_1 \downarrow \\
p_2 \downarrow \downarrow \epsilon \\
G_0 \downarrow \downarrow L_0
\end{array}
\]

We will show first that the map \( sp_1 : L_1 \times_{L_0} G_0 \to L_0 \) is a \( G \)-torsor. We have an action

\[
G_1 \times_{L_0} (L_1 \times_{L_0} G_0) \to L_1 \times_{L_0} G_0
\]

given by \((g, (l, x))) \mapsto (\epsilon(g)^{-1}l, s(g))\) where \( t(b) = \epsilon(x) \). The anchor map \( p_2 : L_1 \times_{L_0} G_0 \to G_0 \) is given by the second projection and \( G_1 \times_{L_0} G_0 \to (L_1 \times_{L_0} G_0) \times_{L_0} (L_1 \times_{L_0} G_0) \) is an homeomorphism since \( \epsilon \) is fully faithful.

Thus, \( sp_1 : L_1 \times_{L_0} G_0 \to L_0 \) is a \( G \)-torsor. Since \( G \) is a stack groupoid, by proposition 6.6 it admits a section. By proposition 5.8 we have that \( \epsilon \) is an equivalence. \( \square \)

Lemma 6.9. \( G \) is a stack groupoid if and only if given an essential equivalence \( \epsilon : A \to B \) and a morphism \( a : A \to G \), there exists a morphism \( \gamma : B \to G \) such that the following diagram commutes up to isomorphism:

\[
\begin{array}{c}
A \ar[r]^a \ar[d]_-\epsilon & G \\
B \ar[ur]^-\gamma
\end{array}
\]

Proof. Consider the weak pushout \( \mathcal{P} \)

\[
\begin{array}{c}
A \ar[r]^a \ar[d]_-\epsilon & G \\
B \ar[r]_-b \ar[u]^-\epsilon' & \mathcal{P}
\end{array}
\]

where \( \epsilon' a \sim b \epsilon \). By stability of essential equivalences under pushout we have that \( \epsilon' \) is an essential equivalence. Now \( \epsilon' : G \to \mathcal{P} \) with \( G \) stack groupoid. From lemma 1 follows that \( \epsilon' \) is an equivalence. Let \( \alpha : \mathcal{P} \to G \) the inverse up to natural transformation: \( \epsilon' \alpha \sim \text{id} \) and \( \alpha \epsilon' \sim \text{id} \). Then take \( \gamma = \alpha b \).

For the reciprocal see Joyal [27]. \( \square \)
7. The homotopy category

**Proposition 7.1.** Let $\epsilon : \mathcal{K} \to \mathcal{G}$ be a functor between fibrant groupoids. Then, $\epsilon$ is an essential equivalence iff it is an equivalence.

**Proof.** By M5 there exists a factorization of $\epsilon$

$$
\begin{array}{c}
\mathcal{K} \\
\downarrow \phi \\
\mathcal{K}'
\end{array}
\xrightarrow{\epsilon} 

\begin{array}{c}
\mathcal{G} \\
\downarrow \varphi
\end{array}
$$

where $\varphi$ is a fibration and $\phi$ is an essential equivalence injective on objects. We have

(1) $\varphi \phi = \epsilon$

and by M2 the map $\varphi$ is an essential equivalence, therefore a trivial fibration and $\varphi$ is an equivalence. Let $\varphi'$ be its inverse up to a natural transformation:

(2) $\varphi \varphi' \sim \text{id}_{\mathcal{G}}$

(3) $\varphi' \varphi \sim \text{id}_{\mathcal{K}'}$

Since $\mathcal{K}$ is fibrant and $\phi$ is a trivial cofibration, there is a lifting $\delta$ that makes the following diagram commutative:

$$
\begin{array}{c}
\mathcal{K} \\
\downarrow \phi \\
\mathcal{K}'
\end{array}
\xrightarrow{\delta} 

\begin{array}{c}
\mathcal{K} \\
\downarrow \text{id}_{\mathcal{K}}
\end{array}
$$

The functor $\delta$ is an essential equivalence by M2 and

(4) $\delta \phi = \text{id}_{\mathcal{K}}$

We will prove that $\delta \varphi'$ is an inverse for $\epsilon$ up to natural transformation.

First, we observe that

(5) $\varphi' \epsilon \sim \phi$

since $\varphi \varphi' \epsilon \sim \text{id}_{\mathcal{G}} \epsilon = \epsilon = \varphi \phi$ using (2), (1) and then proposition 5.4. Then $\delta \varphi'$ is a left inverse up to natural transformation: $\delta \varphi' \epsilon \sim \delta \phi = \text{id}_{\mathcal{K}}$ using (5) and (4). Moreover $\delta \varphi' \epsilon \delta \varphi' \sim \delta \phi \delta \varphi' = \delta \varphi'$ using (5) and (4) again and then since $\delta \varphi'$ is an essential equivalence, we have that $\epsilon \delta \varphi' \sim \text{id}_{\mathcal{G}}$ by proposition 5.4. Therefore, $\epsilon$ is an equivalence with $\delta \varphi'$ its inverse up to natural transformation. \(\square\)

The folk model structure in $\text{Gpd}$ has a homotopy category $\text{HoGpd} = \text{OrbGrd}$ where the morphisms are isomorphism classes of functors (two morphisms are isomorphic if there exists a natural transformation between them).
Corollary 7.2. Let \( \text{OrbGrd} \) be the category of orbifold groupoids with our model structure. The classical homotopy category of \( \text{OrbGrd} \), denoted \( \pi \text{OrbGrd}_{cf} \), is the category whose objects are fibrant groupoids and whose morphisms are isomorphism classes of functors.

Proposition 7.3. The homotopy category of \( \text{OrbGrd} \) is the category \( \text{Orb} = \text{OrbGrd} [E^{-1}] \).

Let \( \text{StOrb} \) denote the classical homotopy category \( \pi \text{OrbGrd}_{cf} \), so we have that the category whose objects are orbifold groupoids is equivalent to the category whose objects are stack groupoids:

Corollary 7.4. \( \text{Orb} \sim \text{StOrb} \)

Appendix: Haefliger’s classifying groupoid

We recall [21] the definition of the classifying space \( BG \) and the universal bundle \( EG \) of a groupoid \( G \).

An element in \( EG \) is a sequence \( (t_0g_0, t_1g_1, \ldots, t_ng_n, \ldots) \), where \( g_i \in G_1 \) are such that \( s(g_i) \) are equal to each other, and \( t_i \in [0, 1] \) are such that all but finitely many of them are zero and \( \sum t_i = 1 \). We set \( (t_0g_0, t_1g_1, \ldots, t_ng_n, \ldots) = (t'_0g'_0, t'_1g'_1, \ldots, t'_ng'_n, \ldots) \) if \( t_i = t'_i \) for all \( i \) and \( g_i = g'_i \) if \( t_i \neq 0 \).

Let \( t_i : EG \to [0, 1] \) denote the map \( (t_0g_0, t_1g_1, \ldots, t_ng_n, \ldots) \mapsto t_i \), and let \( g_i : t_i^{-1}(0, 1) \to G_1 \) denote the map \( (t_0g_0, t_1g_1, \ldots, t_ng_n, \ldots) \mapsto g_i \). The topology on \( EG \) is the weakest topology in which \( t_i^{-1}(0, 1) \) are all open and \( t_i \) and \( g_i \) are all continuous.

The classifying space \( BG \) is the quotient of \( EG \) under the following equivalence relation:

\[
(t_0g_0, t_1g_1, \ldots, t_ng_n, \ldots) \sim (t'_0g'_0, t'_1g'_1, \ldots, t'_ng'_n, \ldots)
\]

if \( t_i = t'_i \) and there is an element \( h \in G_1 \) such that \( g'_i = hg_i \) for all \( i \).

Consider the diagonal action of \( G \) on \( EG \times_{G_0} EG \):

\[
k \cdot ((\ldots, t_ng_n, \ldots), (\ldots, t_nh_n, \ldots)) = ((\ldots, t_ng_n, \ldots), (\ldots, t_nh_n, \ldots))
\]

and take the quotient by this action \( E^* = (EG \times_{G_0} EG) / G \).

The Haefliger’s classifying groupoid \( H\mathcal{G} \) is the following groupoid:

\[
E^* \rightarrow \rightarrow BG
\]

where \( s([(t_0g_0, \ldots, t_ng_n, \ldots)], (t_0h_0, \ldots, t_nh_n, \ldots)]) = (t_0g_0, \ldots, t_ng_n, \ldots) \) and \( t([(t_0g_0, \ldots, t_ng_n, \ldots)], (t_0h_0, \ldots, t_nh_n, \ldots)]) = (t_0h_0, \ldots, t_nh_n, \ldots) \).

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