Convergence Tests

Math 208
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- If $\sum a_i$ converges and c is constant, then $\sum c\, a_i$ converges and equals $c \sum a_i$. If $\sum a_i$ diverges and $c \neq 0$, then $\sum c\, a_i$ diverges.

- If $\sum a_i$ and $\sum b_i$ converge, then $\sum (a_i + b_i)$ converges and equals $\sum a_i + \sum b_i$. The sum of a convergent and divergent series must diverge, but two divergent series can have a sum that converges or diverges (as in 12.1 # 33).

- **Geometric Series:** A geometric series is one where each term is a constant multiple of the previous term. We usually use a for the first term and r for the ratio, writing $\sum_{i=0}^{\infty} ar^i$ or $\sum_{i=1}^{\infty} ar^{i-1}$ or $a + ar + ar^2 + \cdots$. This series converges to $\dfrac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$.

- **$\mathbf{i^{th}}$ Term Test for Divergence:** (also see note below!)
  - If $\sum a_i$ converges, then $\lim_{i\to\infty} a_i = 0$.
  - If $\lim_{i\to\infty} a_i = 0$ is false (i.e. $\lim_{i\to\infty} a_i$ does not exist, or exists and is not zero), then $\sum a_i$ diverges.
  - Nothing can be concluded by only knowing $\lim_{i\to\infty} a_i = 0$.

- **(Direct) Comparison Test:** Suppose $|a_i| \leq b_i$ (for all $i > A$).
  - If $\sum b_i$ converges, then $\sum a_i$ converges.
  - If $\sum a_i$ diverges, then $\sum b_i$ diverges.

- **Ratio Comparison Test:** Suppose $a_i \geq 0$ and $b_i \geq 0$.
  - If $\lim\limits_{i\to\infty} \dfrac{a_i}{b_i} = 0$ and $\sum b_i$ converges, then $\sum a_i$ converges. (Think: $\lim\limits_{i\to\infty} \dfrac{a_i}{b_i} = 0$ says a_i is smaller than b_i.)
  - If $\lim\limits_{i\to\infty} \dfrac{a_i}{b_i} = \infty$ and $\sum b_i$ diverges, then $\sum a_i$ diverges. (Think: $\lim\limits_{i\to\infty} \dfrac{a_i}{b_i} = \infty$ says a_i is bigger than b_i.)
  - If $\lim\limits_{i\to\infty} \dfrac{a_i}{b_i} = L$ with $0 < L < \infty$, then $\sum a_i$ and $\sum b_i$ either both converge or both diverge. ($\lim\limits_{i\to\infty} \dfrac{a_i}{b_i} = L$ says $a_i \approx L\, b_i$.)

- **Alternating Series Test:** A series converges if the following three conditions are met. If all these conditions are not met, nothing can be concluded.
  - $\sum a_i$ is alternating
  - $\lim_{i\to\infty} a_i = 0$ equals zero (also see note below!)
  - $|a_1| \geq |a_2| \geq |a_3| \geq |a_4| \geq \cdots$

- **Integral Test:** If $f(x)$ is positive (i.e. above the x-axis) and decreasing, and we set $a_i = f(i)$, then $\int_1^\infty f(x)\,dx$ converges if and only if $\sum_{i=1}^\infty a_i$ converges.

- **p-Series Test:** The series $\sum \frac{1}{i^p}$ converges if $p > 1$ and diverges if $p \leq 1$. The special case $p = 1$ is the **Harmonic Series**.

- **Ratio and Root Tests:**
  Compute $\lim\limits_{i\to\infty} \left|\frac{a_i}{a_{i-1}}\right|$ (**Ratio Test**) or $\lim\limits_{i\to\infty} \sqrt[i]{|a_i|}$ (**Root Test**).
  - If the limit exists and is less than 1, then $\sum a_i$ converges absolutely.
  - If the limit is infinity or exists and is greater than 1, then $\sum a_i$ diverges.
  - If the limit doesn't exist or exists and equals 1, then nothing can be concluded.
  - *Optional:* For the power series $\sum a_i(x - x_0)^i$ with coefficients a_i, the radius of convergence is $R = 1/L$, where L is the limit above. (Use $1/0 = \infty$ and $1/\infty = 0$.) So the series converges absolutely for $|x - x_0| < R$, diverges for $|x - x_0| > R$, and anything can happen when $|x - x_0| = R$.

Notes:

- $\sum a_i$ **converges absolutely** if $\sum a_i$ and $\sum |a_i|$ both converge.

- $\sum a_i$ **converges conditionally** if $\sum a_i$ converges but $\sum |a_i|$ diverges.

- $\lim_{i\to\infty} a_i = 0$ is equivalent to $\lim_{i\to\infty} |a_i| = 0$, as discussed in class. This can be useful in using the i^{th} Term Test for Divergence or the Alternating Series Test.

- If $\sum |a_i|$ converges, then automatically $\sum a_i$ converges by the Direct Comparison Test.