L-CURVE AND CURVATURE BOUNDS
FOR TIKHONOV REGULARIZATION

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Abstract. The L-curve is a popular aid for determining a suitable value of the regularization parameter when solving linear discrete ill-posed problems by Tikhonov regularization. However, the computational effort required to determine the L-curve and its curvature can be prohibitive for large-scale problems. Recently, inexpensive computable approximations of the L-curve and its curvature, referred to as the L-ribbon and the curvature-ribbon, respectively, were proposed for the case when the regularization operator is the identity matrix. This note discusses the computation and performance of the L- and curvature-ribbons when the regularization operator is an invertible matrix.

Key words. Ill-posed problem, regularization, L-curve.

1. Introduction. Let

\[ Ax = b, \quad A \in \mathbb{R}^{m \times n}, \quad x \in \mathbb{R}^{n}, \quad b \in \mathbb{R}^{m}, \]

be a linear system of equations with a matrix of ill-determined rank, i.e., the singular values of \( A \) "cluster" at the origin. Some singular values may vanish, and the system may be inconsistent. Linear systems of this kind commonly are referred to as linear discrete ill-posed problems. They arise, for instance, when discretizing linear ill-posed problems, such as Fredholm integral equations of the first kind with a smooth kernel. The right-hand side \( b \) in linear discrete ill-posed problems that arise in science and engineering typically is contaminated by an error \( e \in \mathbb{R}^{m} \), which may stem from measurement or discretization errors. In the present paper, we assume that neither the vector \( e \) nor its norm are known.

Let \( b \) be the unknown error-free right-hand side associated with \( b \), i.e.,

\[ b = \tilde{b} + e, \]

and assume that \( \tilde{b} \) is in the range of \( A \). Given the linear system (1) with contaminated right-hand side \( b \), we would like to compute an approximate solution of the consistent linear system

\[ Ax = \tilde{b} \]

with the error-free unknown right-hand side. We denote the solution of (2) of minimal Euclidean norm by \( \tilde{x} \). Since the matrix \( A \) is severely ill-conditioned, the least-squares solution of minimal Euclidean norm of (1) typically is not a meaningful approximation of \( \tilde{x} \).

Tikhonov regularization is a popular approach to remedy this difficulty. Instead of solving the given linear system (1), one determines the solution of the minimization problem

\[ \min_{x \in \mathbb{R}^{n}} \{ \| Ax - b \|^2 + \mu \| Mx \|^2 \}, \]

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where $M$ is referred to as the regularization operator and $\mu \geq 0$ as the regularization parameter. Throughout this paper $\| \cdot \|$ denotes the Euclidean norm.

We would like to determine a value of $\mu \geq 0$, so that the minimal-norm solution $x_\mu$ of (3) is an accurate approximation of the minimal-norm solution $\hat{x}$ of (2). A popular aid for determining such a value of $\mu$, when the norm of the error $e$ is not explicitly known, is the curve

$$
\mathcal{L} := \{ \log \| Mx_\mu \|, \log \| Ax_\mu - b \| : \mu \geq 0 \}.
$$

This curve is commonly referred to as the L-curve, because for many linear discrete ill-posed problems with a right-hand side contaminated by error, the graph of the curve looks like the letter “L.” Hansen [10, 11] proposes to choose the value of the parameter $\mu$ that corresponds to the point at the “vertex” of the “L,” where the vertex is defined to be the point on the L-curve with curvature $\kappa_\mu$ of largest magnitude. We refer to Hansen [10, 11] for a motivation and insightful discussions on properties of solutions $x_\mu$ of (3) determined in this manner.

This approach to determine a suitable value of the regularization parameter typically requires that one computes many points on the L-curve and evaluates the curvature of the L-curve at these points. However, the computational effort required can be prohibitive for large-scale problems. For this reason, the determination of approximations of the L-curve and its curvature, that are inexpensive to compute for large problems, has received considerable attention; see, e.g., [3, 4, 5, 6]. These references are concerned with the case when the regularization operator $M$ is the identity.

For many linear systems of the form (1) with a matrix of ill-determined rank and a contaminated right-hand side a better approximation of the solution $\hat{x}$ of the associated error-free system (2) can be computed by using a regularization operator $M$ different from the identity. This note generalizes the methods developed in [3, 4] to allow invertible regularization operators $M \in \mathbb{R}^{n \times n}$.

The generalization is determined by first transforming the minimization problem to standard form

$$
\min_{x \in \mathbb{R}^n} \{ \| \tilde{A}x - b \|^2 + \mu \| x \|^2 \},
$$

where

$$
\tilde{A} = AM^{-1}, \quad x = Mx.
$$

The formulas of [3, 4] are then applied to the transformed problem (5), and the computed quantities are transformed back to the variables in (3).

Numerical examples indicate that the matrix $M$ may be chosen to be quite ill-conditioned. The restriction that $M$ be invertible, therefore, does not appear to reduce the applicability of the method discussed significantly. In applications where it would be desirable to use a singular $M$, we instead use a near-singular operator. The invertibility of $M$ simplifies the numerical methods significantly. We refer to Björck and Eldén [2, 8] and Hansen [10] for insightful discussions on the choice of and computation with regular and singular regularization operators $M$.

Section 2 discusses the L-curve and its curvature. The formulas differ from those presented in [4] because of the presence of the operator $M$ and because the regularization parameter in the minimization problem considered in that reference is $\mu^2$ instead of $\mu$. Section 3 presents formulas for our approximations of the L-curve and its curvature, referred to as the L-ribbon and the curvature-ribbon respectively. The
L-ribbon is an inexpensively computable set that contains the L-curve in its interior; similarly, the curvature-ribbon is a cheaply computable set that contains the graph of the curvature of the L-curve as a function of $\mu$. The computation of these sets is based on partial Lanczos bidiagonalization of the matrix $\tilde{A}$. Most details will be omitted; they can be derived using results in [3, 4]. Section 4 displays a few computed examples.

2. The L-curve. Since the regularization operator $M$ is nonsingular, the minimization problem (3) has the unique solution

$$x_\mu = (A^T A + \mu M^T M)^{-1} A^T b,$$

for any $\mu > 0$. We will throughout this paper assume that $\mu$ is positive. Introduce

$$\eta_\mu := \|Mx_\mu\|^2; \quad \rho_\mu := \|Ax_\mu - b\|^2.$$

It is easy to show that

$$\eta_\mu = \|x_\mu\|^2 = b^T \tilde{A}(A^T\tilde{A} + \mu I)^{-2} A^T b,$$

$$\rho_\mu := \|Ax_\mu - b\|^2 = \mu^2 b^T (A A^T + \mu I)^{-2} b.$$

Let

$$\bar{\eta}_\mu := \log \eta_\mu, \quad \bar{\rho}_\mu := \log \rho_\mu.$$

Then the point on the L-curve associated with the value $\mu$ of the regularization parameter is given by $P_\mu = \frac{1}{2}(\bar{\eta}_\mu, \bar{\rho}_\mu)$. The curvature of the L-curve at $P_\mu$ is

$$\kappa_\mu = 2 \frac{\rho_\mu' \eta_\mu'' - \rho_\mu'' \eta_\mu'}{(\rho_\mu')^2 + (\eta_\mu')^2)^{3/2}},$$

where $'$ denotes differentiation with respect to $\mu$. It follows from (10) and $\rho_\mu' = -\mu \eta_\mu'$ that

$$\kappa_\mu = 2 \frac{\eta_\mu \rho_\mu \mu \eta_\mu' \rho_\mu + \eta_\mu \rho_\mu + \mu^2 \eta_\mu \eta_\mu'}{(\mu^2 \eta_\mu^2 + \rho_\mu^2)^{3/2}}.$$

3. Lanczos bidiagonalization and the L- and curvature-ribbons. Application of $\ell$ steps of the Lanczos bidiagonalization algorithm bidiag1 by Paige and Saunders [12] to the matrix $\tilde{A} = AM^{-1}$ with initial vector $b$ yields the decompositions

$$AW_\ell = UC_\ell + \delta_{\ell+1}u_{\ell+1}e_1^T, \quad A^T U_\ell = M^T MW_\ell C_\ell^T, \quad b = \delta_1 U_\ell e_1,$$

where the matrices $U_\ell \in \mathbb{R}^{m \times \ell}$ and $W_\ell \in \mathbb{R}^{n \times \ell}$ satisfy $U_\ell^T U_\ell = I_\ell$ and $W_\ell^T M W_\ell = I_\ell$, and the unit vector $u_{\ell+1} \in \mathbb{R}^m$ is such that $U_\ell^T u_{\ell+1} = 0$. The matrix $C_\ell \in \mathbb{R}^{\ell \times \ell}$ is lower bidiagonal, and the $\delta_j$ are nonnegative scalars. Moreover, $I_\ell$ denotes the $\ell \times \ell$ identity matrix and $e_j$ the $j$th axis vector. The matrix-vector product evaluations with the matrix $\tilde{A}$ are computed by first solving a linear system of equations with the matrix $M$, and then multiplying the computed solution by $A$.

For future reference, we note that

$$\text{range}(W_\ell) = \text{span}\{M^{-1} A^T b, M^{-1} A^T A A^T b, \ldots, M^{-1}(A^T A)^{\ell-1} A^T b\}.$$
depends on the choice of regularization operator $M$.

Let $C_{\ell+1} \in \mathbb{R}^{(\ell+1) \times \ell}$ be the bidiagonal matrix, whose leading $\ell \times \ell$ submatrix is $C_{\ell}$ and whose last row is $\delta_{\ell+1}^T e_{\ell}$, where both $C_{\ell}$ and $\delta_{\ell+1}$ are defined by (13). The QR-factorization of $C_{\ell+1} \in \mathbb{R}^{(\ell+1) \times \ell}$ yields an orthogonal matrix and an upper bidiagonal $\ell \times \ell$ matrix. Denote the latter matrix by $\hat{C}_{\ell}$. Given $C_{\ell+1,\ell}$, the matrix $\hat{C}_{\ell}$ can be computed in only $O(\ell)$ arithmetic operations. Let $\hat{C}_{\ell,\ell-1}$ denote the leading $\ell \times (\ell-1)$ submatrix of $\hat{C}_{\ell}$.

We are in a position to present upper and lower bounds of $\rho_\mu$, $\eta_\mu$ and the derivative $\eta'_\mu$ of $\eta_\mu$. It follows from (8) that

\begin{equation}
\eta'_\mu = -2b^T A(A^T A + \mu I)^{-2} A^T b.
\end{equation}

The bounds, denoted by $\rho^-_\mu$, $\eta^-_\mu$ and $(\eta^-_\mu)^+$, respectively, are derived by substituting the Lanczos decomposition (13) into the right-hand sides of (8), (9) and (15), using (6). The connection between the Lanczos decomposition (13) and Gauss quadrature applied to judiciously chosen matrix functionals can be used to show that the derived quantities indeed are upper and lower bounds. We omit the details since they are closely related to those presented in [3, 4]. We obtain

\begin{equation}
\rho^-_\mu \leq \rho_\mu \leq \rho^+_\mu, \quad \eta^-_\mu \leq \eta_\mu \leq \eta^+_\mu, \quad (\eta^-_\mu)^- \leq \eta'_\mu \leq (\eta^-_\mu)^+,
\end{equation}

where

\begin{equation}
\begin{aligned}
\rho^-_\mu &:= \mu^2 \| b \|^2 e_1^T (C_{\ell} C_{\ell}^T + \mu I_{\ell})^{-2} e_1, \\
\rho^+_\mu &:= \mu^2 \| b \|^2 e_1^T (C_{\ell+1,\ell} C_{\ell+1,\ell}^T + \mu I_{\ell+1})^{-2} e_1, \\
\eta^-_\mu &:= \| A^T b \|^2 e_1^T (C_{\ell} C_{\ell}^T + \mu I_{\ell})^{-2} e_1, \\
\eta^+_\mu &:= \| A^T b \|^2 e_1^T (C_{\ell+1,\ell} C_{\ell+1,\ell}^T + \mu I_{\ell+1})^{-2} e_1, \\
(\eta^-_\mu)^- &:= -2 \| A^T b \|^2 e_1^T (C_{\ell,\ell-1} C_{\ell,\ell-1}^T + \mu I_{\ell})^{-3} e_1, \\
(\eta^-_\mu)^+ &:= -2 \| A^T b \|^2 e_1^T (C_{\ell} C_{\ell}^T + \mu I_{\ell})^{-3} e_1.
\end{aligned}
\end{equation}

We evaluate the bounds (17) by solving least-squares problems. For instance, the vector $y_\mu := (C_{\ell} C_{\ell}^T + \mu I_{\ell})^{-1} e_1$, needed for the evaluation of $\rho^-_\mu$, is computed as the solution of the least-squares problem

\[
\min_{y \in \mathbb{R}^\ell} \left\| \begin{bmatrix} C_{\ell}^T \\ \mu^{1/2} I_{\ell} \end{bmatrix} y - \mu^{-1/2} \begin{bmatrix} 0 \\ e_1 \end{bmatrix} \right\|, \quad 0, e_1 \in \mathbb{R}^\ell.
\]

We determine the solution by QR-factorization of the matrix. Since the $C_{\ell}$ is bidiagonal, the solution $y_\mu$ of the least-squares problem can be evaluated in only $O(\ell)$ arithmetic floating point operations for each value of $\mu$; see, e.g., Eldén [8] for details. The other bounds can be computed similarly.

Introduce

\[
\hat{\rho}^-_\mu := \log \rho^-_\mu, \quad \hat{\rho}^+_\mu := \log \rho^+_\mu, \quad \hat{\eta}^-_\mu := \log \eta^-_\mu, \quad \hat{\eta}^+_\mu := \log \eta^+_\mu,
\]

for $\mu > 0$. We define the L-ribbon as the union of the rectangular regions with vertices

\[
\frac{1}{2}(\hat{\eta}^-_\mu, \hat{\rho}^-_\mu), \quad \frac{1}{2}(\hat{\eta}^+_\mu, \hat{\rho}^+_\mu),
\]

\begin{equation}
\bigcup_{\mu > 0} \{ \frac{1}{2}(\hat{\eta}, \hat{\rho}) : \hat{\eta} \leq \hat{\eta} \leq \hat{\eta}^+_\mu, \hat{\rho}^-_\mu \leq \hat{\rho} \leq \hat{\rho}^+_\mu \}.
\end{equation}
We turn to the bounds for the curvature $\kappa_\mu$ of the L-curve. Our bounds are based on (12) and (17). Define the auxiliary quantities

$$
\tau_\mu := 2 \frac{\eta_\mu \rho_\mu}{(\rho^2 \eta_\mu^2 + \rho^2 \eta_\mu^2)^{3/2}}, \\
\xi_\mu := \mu \rho_\mu + \mu^2 \eta_\mu + \rho_\mu \frac{\eta_\mu}{\eta_\mu}.
$$

Their substitution into (12) yields

(19) \hspace{1cm} \kappa_\mu = \tau_\mu \xi_\mu.

Assume that $\rho_\mu^0 \geq 0$, $\eta_\mu^0 \geq 0$, $(\eta_\mu^0)^+ < 0$ and $\mu^2 (\eta_\mu^0)^2 + (\rho_\mu^0)^2 > 0$. Let

$$
\tau_\mu^- := 2 \frac{\eta_\mu^- \rho_\mu^-}{(\mu^2 (\eta_\mu^-)^2 + (\rho_\mu^-)^2)^{3/2}}, \\
\xi_\mu^- := \mu \rho_\mu^- + \mu^2 \eta_\mu^- + \rho_\mu^- \frac{\eta_\mu^-}{\eta_\mu^-}, \\
\tau_\mu^+ := 2 \frac{\eta_\mu^+ \rho_\mu^+}{(\mu^2 (\eta_\mu^+)^2 + (\rho_\mu^+)^2)^{3/2}}, \\
\xi_\mu^+ := \mu \rho_\mu^+ + \mu^2 \eta_\mu^+ + \rho_\mu^+ \frac{\eta_\mu^+}{\eta_\mu^+}.
$$

and

(20) \hspace{1cm} \kappa_\mu^- := \begin{cases} \\
\tau_\mu^- \xi_\mu^-, & \text{if } \xi_\mu^- \geq 0, \\
\tau_\mu^+ \xi_\mu^-, & \text{if } \xi_\mu^- < 0,
\end{cases} \\
(21) \hspace{1cm} \kappa_\mu^+ := \begin{cases} \\
\tau_\mu^- \xi_\mu^+, & \text{if } \xi_\mu^+ \geq 0, \\
\tau_\mu^+ \xi_\mu^+, & \text{if } \xi_\mu^+ < 0,
\end{cases}

Then, similarly as in [4], it can be shown that $0 \leq \tau_\mu^- \leq \tau_\mu \leq \tau_\mu^+$, $\xi_\mu^- \leq \xi_\mu \leq \xi_\mu^+$ and it follows that

(22) \hspace{1cm} \kappa_\mu^- \leq \kappa_\mu \leq \kappa_\mu^+.

The curvature $\kappa_\mu$ as defined by (11) is negative at the “vertex” of the L-curve. We therefore, generally, find it more pleasing to consider $-\kappa_\mu$ instead. The curvature-ribbon, which contains the graph $\{(\mu, -\kappa_\mu) : \mu > 0\}$, is defined as the union of intervals with endpoints (22),

(23) \hspace{1cm} \bigcup_{\mu > 0} \{(\mu, -\kappa) : -\kappa_\mu^+ \leq -\kappa \leq -\kappa_\mu^-\}

The following algorithm determines rectangles associated with the L-ribbon and intervals associated with the curvature-ribbon for the parameter values $\mu_j$, $1 \leq j \leq p$.

**Algorithm 1 (L and Curvature-Ribbon Algorithm).**

**Input:** $b \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $M \in \mathbb{R}^{n \times n}$, $\ell$, $\{\mu_j\}_{j=1}^p$;

**Output:** $\{\rho_\mu^\pm\}_{j=1}^p$, $\{\eta_\mu^\pm\}_{j=1}^p$, $\{\eta_\mu^\pm\}_{j=1}^p$, $\{\kappa_\mu^\pm\}_{j=1}^p$,

i) Compute the entries of the bidiagonal matrix $C_{\ell+1, \ell}$. Determine $\hat{C}_\ell$ by QR-factorization of $C_{\ell+1, \ell}$.

ii) for $j = 1, 2, \ldots, p$ do

Evaluate the bounds $\rho_\mu^\pm$, $\eta_\mu^\pm$ and $(\eta_\mu^\pm)^\pm$ given by (17);

Evaluate the bounds $\kappa_\mu^\pm$ given by (20) and (21);

end j
We conclude this section by noting that an approximation $x_{\mu^*, \ell}$ of the solution $x_{\mu^*}$ of (3) for $\mu = \mu^*$ that corresponds to a point in the L-ribbon (18), can be easily evaluated using the partial Lanczos bidiagonalization (13). Consider the Galerkin equation

$$W^T_\ell (A^T A + \mu^* M^T M) W_\ell y = W^T_\ell A^T b,$$

which, using (13), can be simplified to

$$(C^T_{\ell+1, \ell} C_{\ell+1, \ell} + \mu^* I_\ell) y = \delta_1 C^T_{\ell} e_1.$$ 

The solution $y_{\mu^*, \ell}$ of (24) can be computed as the solution of the least-squares problem

$$\min_{y \in R^\ell} \left\| \begin{bmatrix} C_{\ell+1, \ell}^{1/2} & \mu^* \frac{1}{\delta_1} \\ 0 & 0 \end{bmatrix} y - \begin{bmatrix} e_1 \\ 0 \end{bmatrix} \right\|, \quad e_1 \in R^{\ell+1}, \quad 0 \in R^\ell, \quad \delta_1 = \|b\|.$$ 

Then $y_{\mu^*, \ell}$ yields the approximation $x_{\mu^*, \ell} = W_\ell y_{\mu^*, \ell} = x_{\mu^*}.$ Note that $x_{\mu^*, \ell}$ is in the range of $W_\ell$, which depends on the choice of regularization operator $M$; cf. (14). Thus, the operator $M$ not only affects how $x$ is measured in the minimization problem (3), but also the space in which an approximate solution of the minimization problem is determined.

It can be shown, similarly as Theorem 5.1 in [3] that

$$\frac{1}{2} \eta^+_\mu^* = \log \|Mx_{\mu^*, \ell}\|, \quad \frac{1}{2} \beta^*_\mu^* = \log \|Ax_{\mu^*, \ell} - b\|.$$ 

4. Computed Examples. We illustrate the techniques presented in the previous sections with two numerical examples. All computations were performed with Matlab, version 5.3, on a personal computer with about 16 significant decimal digits.

![Fig. 1. Example 4.1: Rectangles of L-ribbons determined with 5 Lanczos bidiagonalization steps: (a) 40 rectangles for logarithmically equispaced values of $\mu$ in the interval $[1 \cdot 10^{-7}, 1 \cdot 10^{-3}]$ for $M = I$, (b) 40 rectangles for logarithmically equispaced values of $\mu$ in the interval $[1 \cdot 10^{-7}, 1 \cdot 10^{3}]$ for $M$ given by (29).]
Example 4.1. Consider the Fredholm integral equation of the first kind,

\[
\int_0^\pi \exp(s \cos(t))x(t) dt = 2 \frac{\sinh(s)}{s}, \quad 0 \leq s \leq \frac{\pi}{2},
\]

with solution

\[
x(t) := \sin(t).
\]

This equation is discussed by Baart [1]. We use the Matlab program \texttt{baart} in the \textsc{Regularization Tools} package by Hansen [9] to discretize the integral equation by a Galerkin method with 200 orthonormal box functions. This gives a nonsymmetric
matrix $A \in \mathbb{R}^{200 \times 200}$ and a right-hand side vector $\mathbf{b} \in \mathbb{R}^{200}$. The matrix $A$ determined is of ill-determined rank and, in particular, it is numerically singular.

Let $t_j := (2j - 1)\pi / 400, 1 \leq j \leq 200$, and define the scaled tabulation of the solution (27),

$$
\hat{x} := \sqrt{\frac{\pi}{200}} [x(t_1), x(t_2), \ldots, x(t_{200})]^T.
$$

This vector is a good approximation of the solution of the linear system of equations determined by the code baart. We consider it the exact solution of the system (2).

The contaminated right-hand side vector $\mathbf{b}$ in (1) is constructed by first determining a vector $\mathbf{e}$ with normally distributed randomly generated components, scaled so that $||\mathbf{e}|| / ||\mathbf{b}|| = 1 \cdot 10^{-3}$, and then letting $\mathbf{b} := \mathbf{b} + \mathbf{e}$. We refer to the vector $\mathbf{e}$ as “noise.”

We consider the regularization operators $M = I$ and

$$
M = \begin{bmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
\vdots & \vdots & \vdots \\
-1 & 2 & -1 \\
-1 & 2 & -1
\end{bmatrix} \in \mathbb{R}^{n \times n}.
$$

The latter regularization operator is a symmetric tridiagonal Toeplitz matrix, which up to a scaling factor approximates the second derivative operator.

We carry out 5 Lanczos bidiagonalization steps with $M = I$. Figure 1(a) displays rectangles of the L-ribbon for obtained for 40 logarithmically equispaced values $\mu_j$ of the regularization parameter in the interval $[1 \cdot 10^{-8}, 1 \cdot 10^{-6}]$. The upper left corner $\mathbf{M}(\tilde{\eta}_{\mu_j}, \tilde{\rho}_{\mu_j})$ of each rectangle is marked by a cross “×.” The point $\frac{1}{2}(\tilde{\eta}_{\mu_j}, \tilde{\rho}_{\mu_j})$ is associated with the Galerkin solution of (24) for $\mu = \mu_j$; cf. (25). When the rectangles are tiny, only the crosses are visible of the rectangles. Points on the L-curve (4) associated with the exact solution (7) for $\mu = \mu_j$ of the minimization problem (3) are marked by circles “o.” The bounds are tighter for large values of $\mu$ than for small values; see [4] for a discussion of this. The bounds typically improve with the number of Lanczos bidiagonalization steps.

Figure 1(b) is analogous to Figure 1(a) and displays rectangles of the L-ribbon when 5 Lanczos bidiagonalization steps have been carried out with the regularization operator (29). The rectangles shown correspond to 40 logarithmically equispaced values $\mu$ in the interval $[1 \cdot 10^{-3}, 1 \cdot 10^{3}]$. Similarly as in Figure 1(a), the upper left corner of each rectangle is marked by “×” and points on the L-curve are marked by “o.”

The curvature-ribbon obtained with 5 Lanczos bidiagonalization steps and $M = I$ is shown by Figure 2(a) for $1 \cdot 10^{-8} \leq \mu \leq 1 \cdot 10^{-6}$. The upper bound, $-\kappa_{\mu}$, of the quantity $-\kappa_{\mu}$ is marked by a dashed curve and the lower bound, $-\kappa_{\mu}$, by a continuous curve. Let $\mu_L$ denote the value of $\mu$ associated with the point at vertex of the L-curve. Figures 1(a) and 2(a) show that $\mu_L \approx 4 \cdot 10^{-6}$.

Figure 2(b) displays the curvature-ribbon for the regularization operator (29) for $\mu$ in the interval $[1 \cdot 10^{-2}, 1 \cdot 10^{3}]$. Similarly as in Figure 2(a) the upper bound of the curvature is marked by a continuous curve, the lower bound by a dashed curve. Figures 1(b) and 2(b) show that value of the regularization parameter, $\mu_L$, associated with the point at the vertex of the L-curve is about 30. The curvature-ribbons typically
are thinner for large values of $\mu$ than for small values, indicating that the computed curvature bounds are tighter for large values of $\mu$.

It is natural to increase the number of Lanczos bidiagonalization steps $\ell$ until the upper and lower curvature bounds are within a prescribed tolerance in an interval around the location of maximum magnitude of the curvature. In the present example, this condition is satisfied when 5 Lanczos bidiagonalization steps have been carried out.

The dashed curve of Figure 3(a) shows the Galerkin solution $x_{\mu^*,5}$ obtained from (24) with $M = I$ and $\mu^* = 4 \cdot 10^{-6}$. Figure 3(a) also displays the scaled discretized solution $\tilde{x}$ of the integral equation (26) with error-free right-hand side (continuous curve). We have $\|Ax_{\mu^*,5} - b\| = 3 \cdot 10^{-3}$ and $\|x_{\mu^*,5} - \tilde{x}\| = 1 \cdot 10^{-1}$.

Figure 3(b) shows the computed solution $x_{\mu^*,5}$ obtained from (24) with $M$ given by (29) and $\mu^* = 30$ (dashed curve). The continuous curve displays $\tilde{x}$. We have $\|Ax_{\mu^*,5} - b\| = 3 \cdot 10^{-3}$ and $\|x_{\mu^*,5} - \tilde{x}\| = 7 \cdot 10^{-3}$. Thus, the regularization operator (29) yields a much better approximation of (28) than $M = I$.

This example illustrates that the L- and curvature-ribbons make it easy to determine an appropriate number of Lanczos bidiagonalization steps and a suitable value of the regularization parameter. It is also evident that the regularization operator given by (29) is more appropriate for the problem at hand than the identity operator.

The choice of a suitable regularization operator $M$ for a linear discrete ill-posed problem is an important but difficult problem. The L- and curvature-ribbons often allow inexpensive comparisons of different regularization operators, and therefore provide a useful aid when seeking to determine an appropriate regularization operator.

![Figure 4](image)

**Fig. 4. Example 4.2: Rectangles of L-ribbons:** (a) $M = I$, 26 Lanczos bidiagonalization steps and 40 logarithmically equispaced values of $\mu$ in the interval $[1 \cdot 10^{-9}, 1 \cdot 10^{-7}]$. (b) $M$ given by (25), 12 Lanczos bidiagonalization steps and 40 logarithmically equispaced values of $\mu$ in the interval $[1 \cdot 10^{-6}, 1 \cdot 10^{-3}]$.

**Example 4.2.** We are concerned with the Fredholm integral equation of the first kind

\[
\int_0^1 k(s,t)x(t)dt = \frac{1}{6}(s^3 - s), \quad 0 \leq s \leq 1,
\]
Fig. 5. Example 4.2: Curvature-ribbons: (a) \( M = I \) and 26 Lanczos bidiagonalization steps. (b) \( M \) given by (32) and 12 Lanczos bidiagonalization steps.

Fig. 6. Example 4.2: The continuous curves in figures (a) and (b) show (31) as a function of \( t \). The dashed curve in figure (a) displays the computed Galerkin solution \( \mathbf{x}_{\mu_{s, 26}} \) determined by (24) with \( M = I \) and \( \mu_{s} = 8 \cdot 10^{-3} \). The dashed curve in figure (b) shows the computed Galerkin solution \( \mathbf{x}_{\mu_{s, 12}} \) determined by (24) with \( M \) given by (32) and \( \mu_{s} = 7 \cdot 10^{-5} \).

where

\[
k(s, t) := \begin{cases} 
  s(t - 1), & s < t, \\
  t(s - 1), & s \geq t.
\end{cases}
\]

This equation is discussed, e.g., by Delves and Mohamed [7, p. 315]. It has solution \( z(t) = t \). We discretize the integral equation by a Galerkin method with 200 orthonormal box functions, using the Matlab code deriv2 in [9], and obtain the symmetric matrix \( A \in \mathbb{R}^{200 \times 200} \) and right-hand side vector \( \mathbf{b} \in \mathbb{R}^{200} \).

Let \( t_j := (2j - 1)/400, 1 \leq j \leq 200 \), and introduce the scaled tabulation of the
L-curve and curvature bounds

solution of (30),

\[
\hat{x} := \frac{1}{200^{1/2}} [x(t_1), x(t_2), \ldots, x(t_{200})]^T.
\]

This vector is a good approximation of the solution of the linear system of equations determined by the code dervi2, and we will consider it the exact solution of (2).

The contaminated right-hand side vector b in (1) is obtained by adding a “noise” vector e to b, i.e., \( b = \hat{b} + e \). Similarly, as in Example 4.1, we let the components of e be normally distributed randomly generated and scaled so that \( \|e\|/\|b\| = 1 \times 10^{-3} \).

We remark that the matrix A is only moderately ill-conditioned; its spectral norm condition number is \( 5 \cdot 10^4 \). Nevertheless, straightforward solution of the linear system of equations (1) gives a computed solution that has little to do with (31) due to propagated errors.

Figure 4 is analogous to Figure 1 and shows rectangles of the L-ribbons. Figure 4(a) displays rectangles determined by 26 Lanczos bidiagonalization steps with \( M = I \) for 40 logarithmically equispaced values \( \mu_j \) of the regularization in the interval \([1 \cdot 10^{-9}, 1 \cdot 10^{-7}]\). The upper left corner of each rectangle is marked by “×” and the points on the L-curve (4) associated with the values \( \mu_j \) of the regularization parameter are marked by “o.” Figure 4(b) shows 40 rectangles of the L-ribbon associated with logarithmically equispaced values \( \mu_j \) of the regularization parameter in the interval \([1 \cdot 10^{-6}, 1 \cdot 10^{-3}]\). The rectangles are determined by 12 Lanczos bidiagonalization steps with the regularization operator

\[
M = \begin{bmatrix}
1 & -1 & 1 & \cdots & -1 & 1 \\
-1 & 1 & -1 & \cdots & \cdots & 1 \\
& \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & -1 & 1 & \cdots & -1 & 1 \cdot 10^{-8}
\end{bmatrix} \in \mathbb{R}^{n \times n}.
\]

This upper bidiagonal regularization operator approximates the first derivative operator up to a scaling factor. Note that \( M^T M \) is numerically singular, but \( M \) is not. It is easier to implement Tikhonov regularization with nonsingular regularization operators than with singular ones.

Figure 5 is analogous to Figure 2. Figure 5(a) shows the curvature-ribbon for \( M = I \) and \( 1 \cdot 10^{-9} \leq \mu \leq 1 \cdot 10^{-3} \) determined by 26 Lanczos bidiagonalization steps. Similarly, Figure 5(b) presents the curvature-ribbon determined by 12 Lanczos bidiagonalization steps for the regularization operator (32) and \( 1 \cdot 10^{-6} \leq \mu \leq 1 \cdot 10^{-3} \). Figures 4 and 5 show that more Lanczos bidiagonalization steps are required for \( M = I \) than for \( M \) given by (32) in order to be able to determine a good approximation of the values \( \mu_L \) of the regularization parameter associated with a point near the “vertex” of the L-curve. Figures 4(a) and 5(a) suggest that \( \mu_L \approx 8 \cdot 10^{-3} \) when \( M = I \), and Figures 4(b) and 5(b) show that \( \mu_L \approx 7 \cdot 10^{-5} \) when \( M \) is defined by (32).

Finally, Figure 6(a) shows the Galerkin solution \( x_{\mu,26} \) obtained from (24) with \( \mu_* = 8 \cdot 10^{-9} \), \( M = I \) and 26 Lanczos bidiagonalization steps (dashed curve). The continuous curve in Figure 6(a) displays \( \hat{x} \) given by (31). We have \( \|Ax_{\mu,26} - b\| = 4 \cdot 10^{-5} \) and \( \|x_{\mu,26} - \hat{x}\| = 1.2 \cdot 10^{-1} \).

Similarly, the dashed curve of Figure 6(b) presents the computed Galerkin solution \( x_{\mu,12} \) obtained from (24) with \( \mu_* = 7 \cdot 10^{-5} \), \( M \) given by (32) and 12 Lanczos bidiagonalization steps. The continuous curve shows \( \hat{x} \) defined by (31). We have \( \|Ax_{\mu,12} - b\| = 4 \cdot 10^{-5} \) and \( \|x_{\mu,12} - \hat{x}\| = 1.1 \cdot 10^{-2} \).
This example shows that a suitable regularization operator can reduce the computational work by requiring fewer Lanczos bidiagonalization steps and give a better approximation of the desired solution.

In summary, the paper illustrates that the L- and curvature-ribbons can be useful aids both for choosing an appropriate value of the regularization parameter and for choosing a suitable regularization operator $M$. The virtues of a particular choice of $M$ are typically apparent already after only a few Lanczos bidiagonalization steps.

REFERENCES